Wahrscheinlichkeitstheorie 1 **Universität Mannheim**
FSS 2021 Prof. L. Döring, Dr. Q. Shi

Sheet 6

For the exercise class 26.04.2021.
Hand in your solutions before 17:00 Saturday 24.04.2021.

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}_0}, \mathbb{P})$ for all the exercises. All the random variables are assumed to be real-valued.

Exercise 1. Read the lecture notes and summarize the optional stopping theorems (we have a few versions) and the convergence theorems about martingales.

Exercise 2. Consider a metric space (E, d). For $x \in E$ and $B \subseteq E$, define

$$d(x, B) := \inf\{d(x, y) \colon y \in B\}.$$

(i) If $d(x, B) = 0$, then $x \in \bar{B}$, the closure of B.

(ii) Suppose that $B \neq \emptyset$. Prove that

$$|d(x, B) - d(y, B)| \leq d(x, y).$$

This is to say, the function $x \mapsto d(x, B)$ is 1-Lipschitz.

(iii) Let $A \subseteq E$ be **closed**. Show that, for any $\epsilon > 0$, the function

$$f_A^\epsilon(x) = (1 - d(x, A)/\epsilon)^+$$

satisfies the following properties:

- $\forall x \in A$, $f_A^\epsilon(x) = 1$
- if $d(x, A) \geq \epsilon$, then $f_A^\epsilon(x) = 0$
- $f_A^\epsilon \to \mathbf{1}_{\{A\}}$ pointwisely
- $f_A^\epsilon \leq 1$
- $f_A^\epsilon \in \mathrm{Lip}_{1/\epsilon}(E)$ and $f_A^\epsilon \in C_b(E)$

Why do we have to assume A to be closed?

Exercise 3. We define a collection of subsets on $\mathbb{N}$:

$$\mathcal{T} := \{A \subseteq \mathbb{N} \colon A = \emptyset \text{ or } \mathbb{N} \setminus A \text{ is finite }\}.$$

(i) Show that $\mathcal{T}$ is a topology.

(ii) Let $A, B \in \mathcal{T}$. Suppose that $A \neq \emptyset$ and $B \neq \emptyset$. Then prove that $A \cap B \neq \emptyset$.

(iii) Show that the topological space $(\mathbb{N}, \mathcal{T})$ is not metrizable. (In fact, if there is a metric d on $\mathbb{N}$, then you can try to find two open balls (with respect to the metric d) that do not satisfy the property in (ii). If $\mathcal{T}$ is the topology generated by all open balls on the metric space $(\mathbb{N}, d)$, then we are in contradiction to (ii)).

Exercise 4. Consider $\mathbb{R}$ endowed with the Lebesgue measure $\texttt{Leb}$. Let L_∞ be the space of measurable functions that are bounded almost everywhere, with the essential supremum of its absolute value as a norm: for any measurable function f,

$$\|f\|_\infty = \inf\{C \geq 0 \colon \texttt{Leb}(f > C) = 0\}.$$

Show that the metric space L^∞ (with metric induced by the norm $\|\cdot\|_\infty$) is not separable. (consider all the indicator function $\mathbf{1}_{(-r,r)}$, $r > 0$)

Exercise 5. Find the weak limit, if exists, of the following sequence.

(i) $\mathbb{P}_n = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{k/n}$, where δ denotes the Dirac measure.

(ii) $\mathbb{P}_n = \mathcal{N}(0, 1/n)$, the normal distribution.