**Wahrscheinlichkeitstheorie 1** | **Universität Mannheim**
FSS 2021 | Prof. L. Döring, Dr. Q. Shi

# Sheet 5

For the exercise class 19.04.2021.
Hand in your solutions before 17:00 Saturday 17.04.2021.

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}_0}, \mathbb{P})$ for all the exercises. All the random variables are assumed to be real-valued.

**Exercise 1.** Show that the following families are uniformly integrable:

(i) a finite family $\{X_1, X_2, \dots X_n\}$ with each $\mathbb{E}[|X_i|] < \infty$.

(ii) a sequence of identically distributed random variables $\{X_i, i \in \mathbb{N}\}$ with $\mathbb{E}[|X_1|] < \infty$.

(iii) Let $(X_i, i \in \mathbb{N})$ be a sequence of i.i.d. random variables with $\mathbb{E}[|X_1|] < \infty$. Let $S_n = \sum_{i=1}^n X_i$ for every $n \in \mathbb{N}$. Then the family $\{\frac{1}{n}S_n, n \in \mathbb{N}\}$ is uniformly integrable.

**Exercise 2.** Let $(U_n, n \geq 1)$ be i.i.d. Bernoulli random variables with $\mathbb{P}(U_n = 1) = p \in (0, 1)$ and $\mathbb{P}(U_n = 0) = 1 - p$. Set $\tau := \inf\{n \geq 1 \colon U_n = 1\}$. Define $X_n := (1 - p)^n \mathbf{1}_{\{\tau > n\}}$.

(i) Show that $(X_n)$ is a martingale (choose a suitable filtration).

(ii) Prove that $X_n \to 0$ a.s..

(iii) Prove that $(X_n, n \geq 1)$ is not uniformly integrable.

**Exercise 3.** Consider a bag that contains red and blue balls. Initially, there are $r > 0$ red balls and $b > 0$ blue balls. Then we do the following operations. At each step, we draw a ball from the bag, note its colour, and then return it to the bag together with a new ball of the same colour. Let $R_n$ be the number of red balls after $n$ such operations, with $R_0 = r$. Let $\mathcal{F}_n = \sigma(R_0, R_1, \dots R_n)$.

(i) Show that $Y_n = R_n/(n+r+b)$ is a $(\mathcal{F}_n)$-martingale which converges almost surely and in $\mathcal{L}^1$.

(ii) Suppose that $r = b = 1$. Let $T \geq 1$ be the number of balls drawn until the first blue ball appears. If we get draw a blue ball as step 1, then $T = 1$. Show that $\mathbb{E}[(T + 2)^{-1}] = 1/4$.

(iii) Suppose that $r = b = 1$. Show that $\mathbb{P}(\sup_{n\geq 1} Y_n > 3/4) \leq 2/3$.

**Exercise 4.** Gambler's ruin
Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{P}(X_1 = 1) = 1 - \mathbb{P}(X_1 = -1) = p$ for some $p \in (0, 1)$, $p \neq 1/2$. Let $a, b \in \mathbb{N}$ with $0 < a < b$. Define $S_0 := a$ and for every $n \geq 1$, $S_n := S_{n-1} + X_n$. Finally, define the following stopping time:

$$T := \inf\{n \geq 0 : S_n = 0 \text{ or } S_n = b\}.$$

We consider the filtration generated by $(X_n)_{n\geq 1}$.

(i) Show that $\mathbb{E}[T] < \infty$.

(ii) Consider for every $n \geq 0$,

$$M_n := \left(\frac{1-p}{p}\right)^{S_n} \quad \text{and} \quad N_n := S_n - n(2p-1).$$

Prove that $(M_n)_{n\geq 0}$ and $(N_n)_{n\geq 0}$ are martingales.

(iii) Deduce the values of $\mathbb{P}(S_T = 0)$ and $\mathbb{P}(S_T = b)$.

(iv) Compute the value of $\mathbb{E}[T]$.

(v) Let $\tau_0 := \inf\{n \geq 0 : S_n = 0\}$ and $\tau_b := \inf\{n \geq 0 : S_n = b\}$. Compute the value of $\mathbb{P}(\tau_0 < \infty)$ and $\mathbb{P}(\tau_b < \infty)$.

(vi) (Optional) Compute $\mathbb{E}[\tau_0]$ when $p < 1/2$.