**Wahrscheinlichkeitstheorie 1** | **Universität Mannheim**
FSS 2021 | Prof. L. Döring, Dr. Q. Shi

# Sheet 3

For the exercise class 22.03.2021.
Hand in your solutions before 17:00 Saturday 20.03.2021.

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}_0}, \mathbb{P})$ for all the exercises. All the random variables are assumed to be real-valued.

**Exercise 1.** Let $S, T$ be $(\mathcal{F}_n)$-stopping times.

(i) show that $\{S \leq T\} \in \mathcal{F}_{S\wedge T}$ and $\{S = T\} \in \mathcal{F}_{S\wedge T}$.

(ii) show that $S + T$ is an $(\mathcal{F}_n)$-stopping time.

**Exercise 2.** Let $T$ be a $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$-stopping time. Suppose that there exists $N \in \mathbb{N}$ and $\varepsilon \in (0, 1)$, such that for every $n \in \mathbb{N}$,

$$\mathbb{P}(T \leq n + N|\mathcal{F}_n) > \varepsilon, \quad \text{almost surely.}$$

(i) Prove that for each $k \in \mathbb{N}$, we have $\mathbb{P}(T > kN) \leq (1 - \varepsilon)^k$.

**Hint:** $\mathbb{P}(T > kN) = \mathbb{P}(T > kN, T > (k - 1)N)$.

(ii) Deduce that $\mathbb{E}[T] < \infty$.

**Exercise 3.** (From Sheet 2; try to do it this time) Recall that a gamma distribution with parameter $c > 0$ and $\theta > 0$ has density:

$$\frac{\theta^c}{\Gamma(c)} x^{c-1} e^{-\theta x} \mathbf{1}_{\{x>0\}}.$$

(i) Let $X, Y$ be two independent exponential random variables with parameter $\theta > 0$ and $Z = X + Y$. Determine the conditional distribution of $X$ given $Z = z$.

(ii) (Optional) Conversely, let $Z$ be a random variable with gamma distribution with parameter $(2, \theta)$, and suppose $X$ is a random variable whose conditional distribution given $Z = z$ is uniform on $[0, z]$, for $z > 0$. Prove that $X$ and $Z - X$ are independent with exponential distribution $\texttt{Exponential}(\theta)$.

**Exercise 4.** Let $(X_n, n \geq 1)$ is a sequence of i.i.d. random variables with $\mathbb{P}(X_n = 1) = 1/2$ and $\mathbb{P}(X_n = -1) = 1/2$. Let $\mathcal{G}_n := \sigma(X_1, X_2, \ldots X_n)$ with $\mathcal{G}_0 := \{\emptyset, \Omega\}$. Let $S_n := \sum_{i=1}^n X_i$ for every $n \in \mathbb{N}_0$.

(i) Prove that $(S_n^2 - n)_{\mathbb{N}_0}$ is a $(\mathcal{G}_n)$-martingale.

(ii) Prove that $(S_n^3 - 3nS_n)_{\mathbb{N}_0}$ is a $(\mathcal{G}_n)$-martingale.

(iii) Find a polynomial $P(s, n)$ with degree 4 on $s$ and degree 2 on $n$, such that $(P(S_n, n))_{n\in\mathbb{N}_0}$ is a $(\mathcal{G}_n)$-martingale.

(iv) Prove that, in general, for a polynomial $P(x, y)$, the process $(P(S_n, n))_{n\in\mathbb{N}_0}$ is a $(\mathcal{G}_n)$-martingale, if

$$P(s+1, n+1) + P(s-1, n+1) = 2P(s, n).$$

(v) Let $\lambda \in \mathbb{R}$. Find a constant $c \in \mathbb{R}$ such that $\exp(\lambda S_n - cn)_{n\in\mathbb{N}}$ is a $(\mathcal{G}_n)$-martingale.

**Exercise 5** (Part of Lecture; watch the video and then try to do it by yourself)**.** Let $(X_n)_{n\in\mathbb{N}_0}$ be a $(\mathcal{F}_n)$-submartingale. Let $T$ be $(\mathcal{F}_n)$-stopping time.

(i) Suppose that $(X_n)_{n\in\mathbb{N}_0}$ is bounded and $T$ is a.s. finite (the meaning is different from "$T$ is a.s. bounded"!). Prove that $\mathbb{E}[X_0] \leq \mathbb{E}[X_T] < \infty$.

(ii) Suppose that there exists a constant $K > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$|X_n(\omega) - X_{n-1}(\omega)| \leq K, \forall n \in \mathbb{N}.$$

We also suppose that $\mathbb{E}[T] < \infty$. Show that $\mathbb{E}[X_0] \leq \mathbb{E}[X_T] < \infty$.

**Hint:** use the bounded stopping time $T \wedge M$ for some constant $M \in \mathbb{N}$ and the dominated convergence theorem.