Wahrscheinlichkeitstheorie 1 | **Universität Mannheim**
FSS 2021 | Prof. L. Döring, Dr. Q. Shi

Sheet 2

For the exercise class 15.03.2021. Hand in your solutions before 17:00 Saturday 13.03.2021.

Exercise 1. Calculate the following conditional expectations and determine the conditional distributions.

(i) $\mathbb{E}\left[(Y-X)^+|X\right]$, where X, Y are independent uniform random variables on $[0,1]$.

(ii) $\mathbb{E}\left[X|X+Y\right]$, where X, Y are independent Poisson random variables with parameters λ and μ respectively.

(iii) $\mathbb{E}\left[X|Y\right]$, where (X, Y) has a joint distribution with density:

$$4y(x-y)e^{-(x+y)}\mathbf{1}_{\{0<y<x\}}.$$

Exercise 2. For $n \in \mathbb{N}$, let $X_n \in \mathcal{L}^1$ be a non-negative integrable random variable. Suppose that $\liminf_{n\to\infty} \mathbb{E}[X_n] < \infty$. Let $\mathcal{G} \subset \mathcal{A}$ be a sub-σ-algebra. Show that $\liminf_{n\to\infty} X_n \in \mathcal{L}^1$ and prove the analogue of the Fatou lemma:

$$\mathbb{E}[\liminf_{n\to\infty} X_n \mid \mathcal{G}] \leq \liminf_{n\to\infty} \mathbb{E}[X_n \mid \mathcal{G}] \quad \mathbb{P}\text{-a.s.}.$$

Exercise 3. (i) Let X, Y be independent with standard normal distribution $\mathcal{N}(0,1)$. Show that, for any $z \in \mathbb{R}$, the conditional distribution $\mathbb{P}(X \in \cdot \mid X+Y=z)$ is $\mathcal{N}(0, z/2)$.

(ii) Let X, Y are independent Poisson random variables with parameters λ and μ respectively. Determine the conditional distribution $\mathbb{P}(X \in \cdot \mid X+Y=n)$ for $n \in \mathbb{N}$.

Exercise 4.

(i) Let X, Y, Z be three random variables such that (X, Z) and (Y, Z) are identically distributed. We denote by $\mathbb{P}_X$ the distribution of X (so $\mathbb{P}_X$ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$: $\mathbb{P}_X$ is the image measure of $\mathbb{P}$ under X, i.e. $\mathbb{P}_X := \mathbb{P} \circ X$).

  (a) Prove that for any measurable function f non-negative or such that $f(X)$ is integrable,

$$\mathbb{E}[f(X)|Z] = \mathbb{E}[f(Y)|Z] \quad \text{almost surely.}$$

  (b) Let g be a measurable function non-negative or such that $g(Z)$ is integrable. Set

$$h_1(X) := \mathbb{E}[g(Z)|X] \quad \text{and} \quad h_2(Y) := \mathbb{E}[g(Z)|Y].$$

  Show that $h_1 = h_2$ $\mathbb{P}_X$-almost surely.

(ii) Let $T_1, T_2, \ldots, T_n$ be i.i.d. integrable random variables and set $S_n = T_1 + \ldots + T_n$.

  (a) Prove that for any $j \in \{1, \ldots, n\}$, $\mathbb{E}[T_j|S_n] = n^{-1}S_n$ almost surely.

  (b) Compute $\mathbb{E}[S_n|T_j]$ for any $j \in \{1, \ldots, n\}$.

Exercise 5. (Optional) Recall that a gamma distribution with parameter $c > 0$ and $\theta > 0$ has density:

$$\frac{\theta^c}{\Gamma(c)} x^{c-1} e^{-\theta x} \mathbf{1}_{\{x>0\}}.$$

(i) Let X, Y be two independent exponential random variables with parameter $\theta > 0$. Check that the sum $Z = X + Y$ has a gamma distribution with parameter $(2, \theta)$. Moreover, determine the regular conditional distribution of X given Z.

(ii) Conversely, let Z be a random variable with gamma distribution with parameter $(2, \theta)$, and suppose X is a random variable whose conditional distribution given $Z = z$ is uniform on $[0, z]$, for $z > 0$. Prove that X and $Z - X$ are independent with exponential distribution $\texttt{Exponential}(\theta)$.