Wahrscheinlichkeitstheorie 1 — **Universität Mannheim**
FSS 2021 — Prof. L. Döring, Dr. Q. Shi

Sheet 1

For the exercise class 08.03.2021.
Hand in your solutions before 17:00 Sunday 07.03.2021.

For any set E, we denote by $\mathcal{P}(E)$ the powerset of E, i.e. the set of all subsets of E. For all the exercises, unless otherwise specified, we work on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and all the random variables and functions are assumed to be real-valued.

Exercise 1 (Theorem 1.81 in Klenke)**.** Consider two measurable spaces $(E, \mathcal{E})$ and $(F, \mathcal{F})$ and a function $f \colon E \to F$.

(i) Show that $\{B \subset F \colon f^{-1}B \in \mathcal{E}\}$ is a sigma-algebra.

(ii) Show that $\sigma(f) := \{f^{-1}B \subset E \colon B \in \mathcal{F}\}$ is a sigma-algebra.

(iii) Let $\mathcal{B} \subset \mathcal{P}(F)$. Then $f^{-1}(\sigma(\mathcal{B})) = \sigma(f^{-1}(\mathcal{B}))$.

Exercise 2. (Important!)[Factorization lemma] Let $X, Y \colon (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be random variables. Show that, if X is $(\sigma(Y), \mathcal{B}(\mathbb{R}))$-measurable, then there exists a measurable function $g \colon (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, such that $X = g(Y)$.

Exercise 3. (Important!) The three questions are independent.

(i) Let X and Y be i.i.d. Bernoulli variables: $\mathbb{P}(X = 1) = 1 - \mathbb{P}(X = 0) = p$ for some $p \in [0, 1]$. We set $Z := \mathbf{1}_{\{X+Y=0\}}$. Compute $\mathbb{E}[X|Z]$ and $\mathbb{E}[Y|Z]$. Are these random variables independent ?

(ii) Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra and $A \in \mathcal{F}$. We consider $B := \{\mathbb{E}[\mathbf{1}_A|\mathcal{G}] = 0\} \in \mathcal{G}$. Show that $B \subset \Omega \setminus A$ almost surely, i.e. $\mathbb{P}(A \cap B) = 0$.

(iii) Let X be a square-integrable random variable and $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra. We define the *conditional variance*

$$\mathrm{Var}(X|\mathcal{G}) := \mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])^2|\mathcal{G}].$$

Prove the following identity:

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|\mathcal{G})] + \mathrm{Var}(\mathbb{E}[X|\mathcal{G}]).$$

Exercise 4. Let X and Y be two integrable random variables.

(i) Assume that $X = Y$ almost surely. Prove that, almost surely, $\mathbb{E}[X|Y] = Y$ and $\mathbb{E}[Y|X] = X$.

(ii) (Optional) Conversely, suppose that X and Y are such that $\mathbb{E}[X|Y] = Y$ and $\mathbb{E}[Y|X] = X$ almost surely. Prove that $X = Y$ almost surely.
Hint: You can consider $\mathbb{E}[(X - Y)\mathbf{1}_{\{X>c, Y\le c\}}] + \mathbb{E}[(X - Y)\mathbf{1}_{\{X\le c, Y\le c\}}]$.
(Recall that for any $x, y \in \mathbb{R}$, $x > y$ is equivalent to $\exists c \in \mathbb{Q}$ such that $x > c \geq y$.)