Week 3

Exercise 1 (Complete the proof of *Behauptung 2.4*)**.** A subset of $\mathbb{R}^2$ is an **elementary** (*elementar*) set if it is the union of a finite number of rectangles (*Rechtecks*). If two sets $A, B \subset \mathbb{R}^2$ are elementary, prove that $A \cup B$ and $A \triangle B$ are also elementary.

Exercise 2. Let $\mathcal{A}$ be a family of sets. Prove that, the follow statements are equivalent:

(i) $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$ and $A \triangle B \in \mathcal{A}$.

(ii) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$ and $A \backslash B \in \mathcal{A}$.

Exercise 3. Prove that, the measure $\hat{m}$ in *Definition 2.6* is well-defined; see *Bemerkung 2.7.1*.

Exercise 4. Let $(A_k)_{k\geq 1}$ be a sequence of elementary sets and $\hat{m}$ as in *Definition 2.6*. Prove the finite sub-additivity for $\hat{m}$: for any $n \in \mathbb{N}$,

$$\hat{m}(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} \hat{m}(A_i).$$

Exercise 5. Find $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$ for the following cases:

(i) Let $E, F \subset \mathbb{R}$. For any $n = 1, 2, \ldots$, set $A_n = \begin{cases} E, & \text{if } n \text{ is odd,} \\ F, & \text{if } n \text{ is even.} \end{cases}$

(ii) $A_n = \begin{cases} (0, 3 + \frac{1}{n}), & \text{if } n \text{ is odd,} \\ (-1 - \frac{1}{n}, 2], & \text{if } n \text{ is even.} \end{cases}$

(iii) $A_n = [\sin(n) - 1, \sin(n) + 1]$.

Exercise 6. Who are they?