Maß- und Integrationstheorie **Universität Mannheim**
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Week 2

Exercise 1. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. Prove the following statements:

$$\limsup_{n\to\infty} a_n < \alpha \quad \Rightarrow \quad \exists n \geq 0, \forall k \geq n, a_k < \alpha. \tag{1}$$
$$\exists n \geq 0, \forall k \geq n, a_k < \alpha \quad \Rightarrow \quad \limsup_{n\to\infty} a_n \leq \alpha. \tag{2}$$
$$\limsup_{n\to\infty} a_n > \alpha \quad \Rightarrow \quad \forall n \geq 0, \exists k \geq n, a_k > \alpha. \tag{3}$$
$$\forall n \geq 0, \exists k \geq n, a_k < \alpha \quad \Rightarrow \quad \limsup_{n\to\infty} a_n \geq \alpha. \tag{4}$$

Write down similar statements for $\liminf_{n\to\infty} a_n$.

Exercise 2. Let $(A_n)_{n\geq 1}$ be a sequence of subsets of $\mathbb{R}$. Prove the following statements:

$$\left(\limsup_{n\to\infty} A_n\right)^c = \liminf_{n\to\infty} (A_n)^c,$$

$$\limsup_{n\to\infty} (A_n \cup B_n) = \limsup_{n\to\infty} A_n \cup \limsup_{n\to\infty} B_n,$$
$$\limsup_{n\to\infty} (A_n \cap B_n) \subset \limsup_{n\to\infty} A_n \cap \limsup_{n\to\infty} B_n.$$

Exercise 3 (Indicator function)**.** Let $A \subset E$. We define a function $\mathbb{1}_A \colon E \to \{0,1\}$ by

$$\mathbb{1}_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

(i) Let $A, B \subset E$. Write $\mathbb{1}_{A\cap B}$ and $\mathbb{1}_{A\cup B}$ in terms of $\mathbb{1}_A$ and $\mathbb{1}_B$.

(ii) Let $(A_n)_{n\geq 1}$ be a sequence of subsets of E. Rewrite $\mathbb{1}_{\bigcap_{n\geq 1} A_n}$ and $\mathbb{1}_{\bigcup_{n\geq 1} A_n}$ in terms of $\mathbb{1}_{A_n}$.

Exercise 4. Let $(A_n)_{n\geq 1}$ be a sequence of subsets of $\mathbb{R}$.

(i) Prove that

$$\mathbb{1}_{\limsup_{n\to\infty} A_n} = \limsup_{n\to\infty} \mathbb{1}_{A_n}.$$
$$\mathbb{1}_{\liminf_{n\to\infty} A_n} = \liminf_{n\to\infty} \mathbb{1}_{A_n}.$$

(ii) Prove that

$$\limsup_{n\to\infty} A_n = \left\{ \Big(\sum_{n\geq 1} \mathbb{1}_{A_n} \Big) = \infty \right\},$$

$$\liminf_{n\to\infty} A_n = \left\{ \Big(\sum_{n\geq 1} \mathbb{1}_{(A_n)^c} \Big) < \infty \right\},$$

Exercise 5 (Cantor Set)**.** The Cantor ternary set $\mathcal{C}$ is created by iteratively deleting the open middle third from a set of line segments. We begin with the closed unit interval $C_0 := [0, 1]$ and let C_1 denote the set obtained by deleting the open middle third interval, i.e.

$$C_1 = [0, 1/3] \cup [2/3, 1].$$

Next, we delete the middle third interval of each subinterval of C_1; so at the second stage, we get

$$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1].$$

We repeat this procedure for each subinterval of C_2 and so on; see Figure.

So we obtain a sequence of compact sets

$$C_0 \supset C_1 \supset C_2 \supset C_3 \supset \cdots .$$

We define the Cantor set $\mathcal{C}$ to be the intersection of all C_k:

$$\mathcal{C} := \bigcap_{k=0}^{\infty} C_k.$$

Prove the following statements:

(i) $\mathcal{C}$ is compact.

(ii) Given any $x, y \in \mathcal{C}$, there exists $z \notin \mathcal{C}$ that lies between x and y, i.e. $x < z < y$.

(iii) $\mathcal{C}$ has no isolated points.

(iv) $\mathcal{C}$ is uncountable and has the cardinality of the continuum.