Sheet 9

For the exercise class 07.11.2019; please hand in your solutions before 01.11.2019.

Exercise 1. Complete the proofs of *Behauptung 4.11 and Behauptung 4.12*.

Exercise 2. (i) We consider $\mathbb{R}$ endowed with the distance $d(x,y) = \mathbb{1}(x \neq y)$. What is the Borel sigma-algebra of the metric space $(\mathbb{R}, d)$? What is the sigma-algebra generated by all open balls (under metric d), i.e.

$$\sigma\left(B(x,r)\colon x \in \mathbb{R}, r > 0\right)?$$

(ii) Let $(\Omega, \mathcal{F})$ be a measurable space. For every $n \geq 1$, let $f_n\colon \Omega \to \mathbb{R}$ be a real-valued $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable function. Show that, the set

$$\{\omega \in \Omega\colon \lim_{n\to\infty} f_n(\omega) \text{ exists } \} \in \mathcal{F}.$$

(iii) Let $(\Omega, \mathcal{F})$ be a measurable space and (X, d) be an arbitrary metric space with Borel sigma-algebra $\mathcal{B}(X)$. For each $n \geq 1$ let $f_n\colon (\Omega, \mathcal{F}) \to (X, \mathcal{B}(X))$ be a measurable function. Suppose that there exists a function $f\colon \Omega \to X$, such that for every $\omega \in \Omega$,

$$\lim_{n\to\infty} f_n(\omega) = f(\omega).$$

Show that f is $(\mathcal{F}, \mathcal{B}(X))$ measurable.

Exercise 3. Let $C := C([0,1], \mathbb{R})$, the space of all real-valued continuous functions on $[0,1]$. The space C is endowed with the uniform metric d:

$$d(f,g) := \sup\{|f(x) - g(x)|\colon x \in [0,1]\}.$$

Let $\mathcal{B}(C)$ be the Borel sigma-algebra of the metric space (C, d). For every $x \in [0,1]$, define a function $\phi_x\colon C \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by

$$\phi_x(f) = f(x).$$

Let $\sigma(f_x, x \in [0,1])$ be the minimum sigma-algebra such that all function $\phi_x, x \in [0,1]$ are measurable; see *Beispiel 4.4 5(b)*. Prove the identity $\mathcal{B}(C) = \sigma(f_x, x \in [0,1])$.

Exercise 4 (Product measurable space)**.** Let $(\Omega_\alpha, \mathcal{F}_\alpha)_{\alpha \in I}$ be a family of measurable spaces. For any collection $A \subset I$, we define a space Ω_A by the Cartesian product

$$\Omega_A := \prod_{\alpha \in A} \Omega_\alpha;$$

i.e. each element in Ω_A is a tuple $(x_\alpha)_{\alpha \in A}$ with $x_\alpha \in \Omega_\alpha$. For any $B \subset A$, we define the projection map $\pi_{A \to B}\colon \Omega_A \to \Omega_B$ by

$$\pi_{A\to B}((x_\alpha)_{\alpha \in A}) = (x_\alpha)_{\alpha \in B}.$$

For simplicity, we denote $\pi_A := \pi_{I\to A}$ and $\pi_\alpha := \pi_{\{\alpha\}}$. Moreover, we define a *product sigma-algebra* $\mathcal{F}_A$ on Ω_A by

$$\mathcal{F}_A := \prod_{\alpha\in A} \mathcal{F}_\alpha := \sigma\left(\pi^{-1}_{A\to\{\alpha\}}(B)\colon B \in \mathcal{F}_\alpha, \alpha \in A\right).$$

Recall from *Beispiel 4.4.5(b)* that $\mathcal{F}_A$ is the smallest sigma-algebra on Ω_A such that $(\pi_{A\to\{\alpha\}}, \alpha \in A)$ are all measurable, i.e. $\mathcal{F}_A = \sigma(\pi_{A\to\{\alpha\}}, \alpha \in A)$.

(i) For any $B \subset A \subset I$, show that $\pi_{A\to B}$ is $(\mathcal{F}_A, \mathcal{F}_B)$-measurable.

(ii) For any $E_I \in \Omega_I$ show that there exists an at most countable set $B \subset I$ and $E_B \in \mathcal{F}_B$, such that $E_I = \pi_B^{-1}(E_B)$.

(iii) If I is at most countable, show that

$$\mathcal{F}_I = \sigma\left(\prod_{\alpha\in I} B_\alpha, B_\alpha \in \mathcal{F}_\alpha\right).$$

Is this still true when I is uncountable?