Maß- und Integrationstheorie **Universität Mannheim**
HWS 2019 Dr. H. Pitters, Dr. Q. Shi

Sheet 8

For the exercise class 31.10.2019; please hand in your solutions before 25.10.2019.

Exercise 1. Consider the integers $\mathbb{Z}$ and define

$$\mathcal{A} := \{A \subset \mathbb{Z} \colon A \text{ is finite or } A^c \text{ is finite } \}.$$

Define a function $\mu \colon \mathcal{A} \to \{0, 1\}$ by

$$\begin{cases} \mu(A) = 0, & \text{if } A \text{ is finite} \\ \mu(A) = 1, & \text{if } A^c \text{ is finite.} \end{cases} \tag{1}$$

Prove that, μ has no extension (*Fortsetzung*) to $\sigma(\mathcal{A})$.

Exercise 2. The purpose of this exercise is to show that the σ-finite hypothesis in *Folgerung 3.33* cannot be removed. In other words, if μ is not sigma-finite, then the extension of μ is NOT unique.

Let $\mathcal{A}$ be the collection of all subsets in $\mathbb{R}$ that can be expressed as **finite** unions of half-open intervals $(a, b]$; thus $\mathcal{A}$ is an algebra. Let $\mu \colon \mathcal{A} \to [0, +\infty]$ be the function such that $\mu(E) = +\infty$ for non-empty E and $\mu(\emptyset) = 0$.

(i) Show that μ satisfies the assumptions in *Theorem 3.31*; such a function is often called a *pre-measure*.

(ii) Show that $\sigma(\mathcal{A})$ is the Borel σ-algebra $\mathcal{B}(\mathbb{R})$.

(iii) Show that the Hahn-Kolmogorov extension $\mu \colon \mathcal{B}(\mathbb{R}) \to [0, +\infty]$ of μ, obtained by using *Theorem 3.31*, assigns an infinite measure to any non-empty Borel set.

(iv) Show that counting measure $\#$ (i.e. $\#(A) = card(A)$ gives the number of elements in A) is another extension of μ on $\mathcal{B}(\mathbb{R})$.

Exercise 3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and suppose that collections of sets $(\mathcal{A}_i \subset \mathcal{F}, i \in [n])$ are independent. Recall that, we say they are independent, if whenever for any $I \subset [n]$ and any $A_i \in \mathcal{A}_i, i \in I$, we have

$$\mathbb{P}(\bigcap_{i \in I} A_i) = \prod_{i \in I} \mathbb{P}(A_i).$$

Furthermore, suppose that each $\mathcal{A}_i$ for $i \in [n]$ is closed under intersection ($\cap$-*stabil*). Prove that, $(\sigma(\mathcal{A}_i), i \in [n])$ are also independent.

Hint: fix any $A_2 \in \mathcal{A}_2, \ldots, A_n \in \mathcal{A}_n$ with $F := A_2 \cap \cdots \cap A_n$, and prove that the following collection is a sigma-algebra:

$$\{A \in \mathcal{F} \colon \mathcal{P}(A \cap F) = \mathcal{P}(A)\mathcal{P}(F)\}.$$