**Maß- und Integrationstheorie** | **Universität Mannheim**
HWS 2019 | Dr. H. Pitters, Dr. Q. Shi

# Sheet 7

For the exercise class 24.10.2019; please hand in your solutions before 18.10.2019.

We denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel sigma-algebra on $\mathbb{R}^d$ (see *Beispiel 3.13*).

**Exercise 1.** Let $\mathcal{A}$ be a collection of subsets of the space $\Omega$ show that the following statements are equivalent:

(i) $\mathcal{A}$ is a Dynkin-system;

(ii) $\mathcal{A}$ satisfies the following properties:

- $\Omega \in \mathcal{A}$;
- if $A, B \in \mathcal{A}$ and $A \subset B$, then $B \setminus A \in \mathcal{A}$;
- For an increasing sequence $(A_n \in \mathcal{A}, n \geq 1)$ with $A_n \subset A_{n+1}$ for every $n \geq 1$, we have $\bigcup_{n \geq 1} A_n \in \mathcal{A}$.

**Exercise 2.** Give an example of a Dynkin-system, that is not a sigma-algebra.

**Exercise 3.** Give an example of two probability measures $\mu \neq \nu$ on $\mathcal{F} := \{$ all subsets of $\{1, 2, 3, 4\}\}$, such that for a certain collection of sets $\mathcal{A} \subset \mathcal{F}$ with $\sigma(\mathcal{A}) = \mathcal{F}$ (i.e. the sigma-algebra generated by $\mathcal{A}$ is $\mathcal{F}$ ), there is the identity $\mu(A) = \nu(A)$ for every $A \in \mathcal{A}$.

**Exercise 4.** Prove that, the the Borel sigma-algebra on $\mathbb{R}^1$ is the sigma-algebra generated by any of the following collection (in the sense of *Definition 3.12*):

(i) all closed subsets of $\mathbb{R}^1$

(ii) all compact subsets of $\mathbb{R}^1$

(iii) all $(a, b)$ with $a, b \in \mathbb{R}^1$

(iv) all $[a, b]$ with $a, b \in \mathbb{R}^1$

(v) all $(a, b]$ with $a, b \in \mathbb{R}^1$

(vi) all $(-\infty, a]$ with $a \in \mathbb{R}^1$

(vii) all $(-\infty, a)$ with $a \in \mathbb{R}^1$

**Exercise 5** (Regularity of a Borel measure)**.** let $\mu$ be any measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and suppose that for every ball with finite radius, say $B(x, r)$ with $x \in \mathbb{R}^d$ and $r \in [0, \infty)$, we have $\mu(B(x, r)) < \infty$. The purpose of this exercise is to prove the following property in a few steps:

For any Borel set $E \in \mathcal{B}(\mathbb{R}^d)$ and any $\epsilon > 0$, there exists $F \subset E \subset O$, with $F$ closed and $O$ open, such that

$$\mu(O \setminus E) < \epsilon \quad \text{and} \quad \mu(E \setminus F) < \epsilon.$$

(i) Let $F_k \subset \mathbb{R}^d, k \geq 1$ be a sequence of closed sets and write $F^* := \bigcup_{k=1}^{\infty} F_k$ (note that $F^*$ is not necessarily closed). Then show that, for any $\epsilon > 0$, there exists a closed set $F \subset F^*$, such that $\mu(F^* \setminus F) < \epsilon$.

(ii) Define $\mathcal{A}$ to be the collection of all sets $E \in \mathcal{B}(\mathbb{R}^d)$ that satisfies the stated property. Prove that, $\mathcal{A}$ is a sigma-algebra.

(iii) Conclude that $\mathcal{A} = \mathcal{B}(\mathbb{R}^d)$.

**Exercise 6.** Let $\mu, \nu$ be two (positive) measures on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

(i) Suppose that for all $a, b \in \mathbb{R}$ with $a < b$, the following inequality holds:

$$\mu((a, b)) \leq \nu((a, b)).$$

Prove that $\mu(A) \leq \nu(A)$ for every $A \in \mathcal{B}(\mathbb{R})$.
**Hint**: first consider an open set $A$ and then use the regularity in Exercise 1.

(ii) Suppose that, for every $a \in \mathbb{R}$, the following inequality holds:

$$\mu((-\infty, a)) \leq \nu((-\infty, a)).$$

Can we deduce the same conclusion as in (i)?