Maß- und Integrationstheorie | **Universität Mannheim**
HWS 2019 | Dr. H. Pitters, Dr. Q. Shi

Sheet 6

For the exercise class 17.10.2019; please hand in your solutions before 11.10.2019.

For any space Y, denote by $\mathcal{P}(Y)$ the ensemble of ALL subsets of Y. For a collection of subsets of Y, $\mathcal{A} \subset \mathcal{P}(Y)$, we denote by $\sigma(\mathcal{A})$ the sigma-algebra generated by $\mathcal{A}$ in the sense of *Definition 3.12*.

Exercise 1. Let $(\Omega, \mathcal{F}, \mu)$ be a measured space (*Massraum*). We write

$$\mathcal{N}_\mu := \Big\{ N \subseteq \Omega \colon \; \exists B \in \mathcal{F}, \; N \subseteq B \text{ and } \mu(B) = 0 \Big\}$$

for the set of μ-negligible (*vernachlässigbar*) subsets of Ω.

(i) We define the completion (*Vervollständigung*) of $\mathcal{F}$ with respect to μ by

$$\mathcal{F}_\mu := \Big\{ A \subseteq \Omega \colon \; \exists E, F \in \mathcal{F}, \; E \subseteq A \subseteq F \text{ and } \mu(F \setminus E) = 0 \Big\}.$$

Prove that $\mathcal{F}_\mu \supseteq \mathcal{F}$ is still a σ-algebra on Ω.

(ii) Show that $\mathcal{F}_\mu = \{A \subseteq \Omega \colon \; \exists (E, N) \in \mathcal{F} \times \mathcal{N}_\mu, \; A = E \cup N\}$.

Exercise 2. Consider any space Ω and $\mathcal{A} \subset \mathcal{P}(\Omega)$. Prove that, for any $B \in \sigma(\mathcal{A})$, there exists a family $\mathcal{D} \subset \mathcal{A}$ of at most countable cardinality, such that $B \in \sigma(\mathcal{A})$.
Hint: consider $\{B \in \sigma(\mathcal{A}) \colon$ there exists $\mathcal{D} \subset \mathcal{A}$ at most countable such that $B \in \sigma(\mathcal{D})\}$ and prove that this is a sigma-algebra.

Exercise 3 (True of False?)**.** Prove or disprove (with a counterexample) the following statements:

(i) Let $\mathcal{E}$ and $\mathcal{F}$ be two sigma-algebras of E. Then the union $\mathcal{E} \cup \mathcal{F}$ is a sigma-algebra.

(ii) Let $\mathcal{F}$ be a sigma-algebra of Ω and $B \subset \Omega$. Then $\mathcal{F}_B := \{A \cap B \colon A \in \mathcal{F}\}$ is a sigma-algebra of B.

(iii) Let $\mathcal{F}$ be a sigma-algebra on $X \times Y$ and $\pi \colon X \times Y \to X$ be the canonical projection. Then $\mathcal{F}_X := \{\pi(F) \colon F \in \mathcal{F}\}$ is a sigma-algebra of X.

(iv) For any $n \in \mathbb{N}$, define a sigma-algebra $\mathcal{F}_n := \sigma(\{1\}, \ldots, \{n\})$ on $\mathbb{N}$. For example, $\mathcal{F}_1 = \{\emptyset, \{1\}, \mathbb{N}, \mathbb{N} \setminus \{1\}\}$. But $\bigcup_{n \geq 1} \mathcal{F}_n$ is NOT a sigma-algebra.

(v) Let X, Y be two spaces and $f \colon X \to Y$ a function. Let $\mathcal{A} \subset \mathcal{P}(Y)$. Then $f^{-1}(\sigma(\mathcal{A})) = \sigma(f^{-1}(\mathcal{A}))$.

Exercise 4 (Bonus; whoever solves this exercise during this semester wins a box of chocolate)**.** Prove *Theorem 2.34* in the script: let $\Omega \subset \mathbb{R}^d$ be a **uncountable** set. Suppose that $\mu \colon \mathcal{P}(\Omega) \to [0, \infty)$ satisfies the following properties:

(i) $\mu(\Omega) < \infty$;

(ii) $\mu(\{x\}) = 0$ for every $x \in \Omega$;

(iii) sigma-additivity: i.e., for any disjoint sequence $(A_i \subset \Omega, i \in \mathbb{N})$, we have the identity

$$\mu(\bigcup_{i\geq 1} A_i) = \sum_{i\geq 1} \mu(A_i).$$

Hint: refer to `https://en.wikipedia.org/wiki/Measurable_cardinal`
and `https://eudml.org/doc/212487`