Maß- und Integrationstheorie **Universität Mannheim**
HWS 2019 Dr. H. Pitters, Dr. Q. Shi

Sheet 5

For the exercise class 10.10.2019; please hand in your solutions before 04.10.2019.

Let λ denote the Lebesgue measure on $\mathbb{R}^d$. We say $A \subset \mathbb{R}^d$ is (Lebesgue)-measurable in the sense of *Definition 2.29*. We denote by $\mathfrak{A}$ the ensemble of Lebesgue-measurable sets in $\mathbb{R}^d$.

Exercise 1 (Regularity of Lebesgue measure)**.** Let $A \in \mathfrak{A}$ be Lebesgue-measurable. Show that

$$\lambda(A) = \sup_{K \subset A, K \text{compact}} \lambda(K).$$

Exercise 2. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with $\lambda(E) > 0$.

a) Prove that for any $\epsilon > 0$, there exists a bounded open interval J such that $\lambda(E \cap J) \geq (1 - \epsilon)\lambda(J)$.

b) Prove that the difference set of E, which is defined by

$$(E - E) := \{z \in \mathbb{R} : z = x - y \text{ for some } x, y \in E\},$$

contains an open interval centered at the origin.

Exercise 3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. This function defines a curve $\Gamma \subset \mathbb{R}^2$:

$$\Gamma := \{(x, y) \in \mathbb{R}^2\colon x \in \mathbb{R}, y = f(x)\}$$

Show that Γ is Lebesgue-measurable and its Lebesgue measure $\lambda(\Gamma) = 0$.

Exercise 4 (True or false?)**.** Let $A \in \mathfrak{A}$ be Lebesgue-measurable. Prove or disprove (with a counter-example) the following statements:

(i) If $B \subseteq A$ then $B \in \mathfrak{A}$.

(ii) If $\lambda(A) = \infty$ then A is an unbounded set.

(iii) If $\lambda(A) < \infty$ then A is a bounded set.

(iv) If $\lambda(A) = 0$ then A is a bounded set.

(v) If A is an open set then $\lambda(A) > 0$.

(vi) If $\lambda(A \cap (0, 1)) = 1$ then $A \cap (0, 1)$ is dense in $(0, 1)$.

(vii) If $A \cap (0, 1)$ is dense in $(0, 1)$ then $\lambda(A \cap (0, 1)) > 0$.

(viii) If $\lambda(A) > 0$ then A has a non-empty interior (*nicht leeres Inneres*).

Exercise 5 (Cardinality of a σ-algebra)**.** Let E be any space and $\mathcal{A}$ be a σ-algebra on E. Suppose that $\mathcal{A}$ has at most countably many elements. For any $x \in E$, define a subset $B_x \subset E$ by

$$B_x := \bigcap_{A \in \mathcal{A}\colon\, x \in A} A.$$

(i) Show that the family $\{B_x, x \in E\}$ yields a partition of E.

(ii) Prove that, $B_x \in \mathcal{A}$ for any $x \in E$. Then justify that, each element $A \in \mathcal{A}$ can be written as the union of a collection of some B_x's.

(iii) Conclude that, E and $\mathcal{A}$ both have finite elements; i.e. $\mathcal{A}$ cannot have the cardinality of the natural numbers.