Maß- und Integrationstheorie | **Universität Mannheim**
HWS 2019 | Dr. H. Pitters, Dr. Q. Shi

Sheet 13

For the exercise class 05.12.2019.

We denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel sigma-algebra on $\mathbb{R}^d$ (see *Beispiel 3.13*). We denote by dx the Lebesgue measure.
a.e. $=$ almost everywhere (*fast überall*)

Exercise 1. Let $f\colon (\Omega, \mathcal{F}, \mu) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a measurable function. Suppose that μ is sigma-finite. Prove that, for Leb-a.e. $y \in \mathbb{R}$, we have

$$\mu(\{f = y\}) = 0, \quad \text{for Leb-a.e. } y \in \mathbb{R}.$$

Exercise 2. Let $f\colon (\Omega, \mathcal{F}, \mu) \to (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ be a non-negative measurable function. Suppose that μ is sigma-finite.

(i) Let $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a C^1-function with $g(0) = 0$. Show that

$$\int_\Omega g \circ f d\mu = \int_0^\infty g'(t)\mu(f \geq t)dt.$$

(ii) Show that

$$\int_\Omega f d\mu = \int_0^\infty \mu(f \geq t)dt.$$

(iii) Suppose that μ is a finite measure and there exists $p \geq 1$ and $c > 0$ such that, for all $t > 0$,

$$\mu(|f| > t) \leq ct^{-p}.$$

Show that, for every $q \in [1, p)$, $\int |f|^q d\mu < \infty$.

Exercise 3. (i) Let $t > 0$. Show that

$$\int_{(0,t)} \frac{\sin x}{x}\, dx = \int_{(0,\infty)} \left(\int_{(0,t)} e^{-xy} \sin x \, dx \right) dy.$$

(ii) Deduce that

$$\int_{(0,t)} \frac{\sin x}{x}\, dx = \int_{(0,\infty)} \frac{1 - e^{-ty}\,(y \sin t + \cos t)}{1 + y^2}\, dy \tag{1}$$

for all $t > 0$, and conclude that

$$\lim_{t \to \infty} \int_{(0,t)} \frac{\sin x}{x}\, dx = \frac{\pi}{2}. \tag{2}$$

(iii) Is the function $x \mapsto \dfrac{\sin x}{x}$ Lebesgue-integrable on $(0, \infty)$?

Exercise 4 (Riesz–Scheffé lemma). Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, and $f, f_1, f_2, \ldots \in L^p(\Omega, \mathcal{A}, \mu)$ with $p \in [1, \infty)$. We suppose that, as $n \to \infty$, $f_n(\omega) \to f(\omega)$ for μ-a.e. $\omega \in \Omega$ and that $\|f_n\|_p \to \|f\|_p$. Let $\chi\colon \mathbb{R} \to \{-1, 1\}$ denote a function such that $|x| = \chi(x)x$ for all $x \in \mathbb{R}$, and write $f_n^* := f_n \mathbb{1}_{\{|f_n| \le |f|\}} + \chi(f_n)|f|\mathbb{1}_{\{|f_n| > |f|\}}$ for every $n \in \mathbb{N}$.

(i) Show that $\|f_n^* - f\|_p \to 0$ as $n \to \infty$.

(ii) Show that $\|f_n - f_n^*\|_p \to 0$ as $n \to \infty$. Conclude that $f_n \to f$ in $L^p(\Omega, \mathcal{A}, \mu)$.
Hint: Use the convexity inequality $(y - x)^p \le y^p - x^p$ for $0 \le x \le y$.