Maß- und Integrationstheorie **Universität Mannheim**
HWS 2019 Dr. H. Pitters, Dr. Q. Shi

Sheet 12

For the exercise class 28.11.2019.

We denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel sigma-algebra on $\mathbb{R}^d$ (see *Beispiel 3.13*). We denote by dx the Lebesgue measure.
a.e. = almost everywhere (*fast überall*)
Let ν and μ be two measures on a measurable space $(\Omega, \mathcal{F})$. Then we say

(i) ν is *absolutely continuous with respect to* μ, denoted by $\nu \ll \mu$, if

$$\forall A \in \mathcal{F}, \qquad \mu(A) = 0 \Rightarrow \nu(A) = 0.$$

(ii) ν and μ are *mutually sigular*, denoted by $\nu \perp \mu$, if there exists $A \in \mathcal{F}$, such that

$$\mu(A) = 0 \quad \text{and} \quad \nu(A^c) = 0.$$

Exercise 1. (i) (Radon–Nikodym) Suppose ν and μ are sigma-finite (positive) measures on a measurable space $(\Omega, \mathcal{F})$. Then there exists a unique couple (ν_a, ν_s) of measures on $(\Omega, \mathcal{F})$, such that

$$\nu = \nu_a + \nu_s, \quad \nu_a \ll \mu, \quad \nu_s \perp \mu.$$

Moreover, there exists a non-negative measurable function $f\Omega \to \mathbb{R}_+$ such that

$$\nu_a(A) = \int_A f d\mu, \quad \forall A \in \mathcal{F}. \tag{1}$$

(ii) (A counter-example for Radon–Nikodym) Let μ be the counting measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, i.e. $\mu(A) = \#A$. Show that, the Lebesgue measure λ is absolutely continuous with respect to μ; but there does NOT exist a measurable function such that a relation as in (1) holds.

Exercise 2 (Conditional expectation)**.** Let X be a non-negative measurable (resp. absolutely integrable) function from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $\mathcal{A} \subset \mathcal{F}$ be a sigma-algebra.

(i) Prove that, there exists a non-negative measurable (resp. absolutely integrable) function Y from the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, such that

$$\int_A X d\mathbb{P} = \int_A Y d\mathbb{P}, \quad \forall A \in \mathcal{A}.$$

Denote by $\mathbb{E}[X|\mathcal{A}] := Y$, which is called *the conditional expectation of X given $\mathcal{A}$*.

(ii) Let X, X' be non-negative measurable functions from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $\alpha, \alpha' \in \mathbb{R}$. Show that

$$\mathbb{E}[\alpha X + \alpha' X' | \mathcal{A}] = \alpha \mathbb{E}[X|\mathcal{A}] + \alpha' \mathbb{E}[X'|\mathcal{A}].$$

(iii) Let X, X' be non-negative measurable functions from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Suppose that X' is $(\mathcal{A}, \mathcal{B}(\mathbb{R}))$-measurable, then show that

$$\mathbb{E}[XX'|\mathcal{A}] = X'\mathbb{E}[X|\mathcal{A}].$$

(iv) Let $\mathcal{A}_1 \subset \mathcal{F}$ and $\mathcal{A}_2 \subset \mathcal{F}$ be two sigma-algebras. Prove that, $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent, if and only if

$$\mathbb{E}[\mathbb{1}_A|\mathcal{A}_1] = \mathbb{P}(A), \quad \text{for every } A \in \mathcal{A}_2.$$