Universität Mannheim
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Sheet 11

For the exercise class 21.11.2019.

We denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel sigma-algebra on $\mathbb{R}^d$ (see *Beispiel 3.13*). We denote by dx the Lebesgue measure.
a.e. = almost everywhere (*fast überall*)

Exercise 1. (i) Let f be a differentiable real-valued function on $[0,1]$ with f' bounded. Show that $\int_{[0,1]} f'(x)dx = f(1) - f(0)$.

(ii) Find a continuous function $f\colon [0,1] \to \mathbb{R}$ that is a.e. differentiable with $\int_{[0,1]} f'(x)dx = 0$.

Exercise 2. Let $f\colon (\Omega, \mathcal{F}, \mu) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$ be an integrable function.

(i) Show that $\lim_{n\to\infty} \int_\Omega |f| \mathbb{1}_{\{|f|>n\}} d\mu = 0$.

(ii) For any $\epsilon > 0$, there exists $\delta > 0$, such that for any $A \in \mathcal{F}$ with $\mu(A) < \delta$, we have $\int_A |f| d\mu < \epsilon$.

Exercise 3 (Integral depending on a parameter)**.** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $(a,b) \subset \mathbb{R}$. For a function $f\colon \Omega \times (a,b) \to \mathbb{R}$, suppose that for every $u \in (a,b)$, the function $x \mapsto f(x,u)$ is $\mathcal{F}, (\mathbb{R}, \mathcal{B}(\mathbb{R}))$-measurable and μ-integrable. Fix $u_0 \in (a,b)$.

(i) Suppose that

(a) for μ-a.e. $x \in \Omega$, the function $u \mapsto f(x,u)$ is continuous at u_0;

(b) there exists a non-negative μ-integrable function g, such that $|f(x,u)| \le g(x)$ μ-a.e..

Then prove that, the function $F(u) := \int_\Omega f(x,u)\mu(dx)$ is continuous at $u_0 \in (a,b)$.

(ii) Suppose that

(a) for μ-a.e. $x \in \Omega$, the function $u \mapsto f(x,u)$ is differentiable at u_0, with the derivative denoted by $\frac{\partial f}{\partial u}(x, u_0)$;

(b) there exists a non-negative μ-integrable function g, such that $|f(x,u) - f(x,u_0)| \le g(x)|u - u_0|$ μ-a.e..

Then prove that, the function $F(u) := \int_\Omega f(x,u)\mu(dx)$ is differentiable at $u_0 \in (a,b)$ and the derivative is

$$F'(u_0) = \int_\Omega \frac{\partial f}{\partial u}(x,u_0)\mu(dx).$$

(iii) The assumptions in (ii) can be replaced by the followings:

(a) for μ-a.e. $x \in \Omega$, the function $u \mapsto f(x,u)$ is differentiable in (a,b), with the derivative denoted by $\frac{\partial f}{\partial u}(x,u)$ for all $u \in (a,b)$;

(b) there exists a non-negative μ-integrable function g, such that for μ-a.e. x,

$$\left|\frac{\partial f}{\partial u}(x,u)\right| \leq g(x), \forall u \in (a,b).$$

Then prove that, the function $F(u) := \int_\Omega f(x,u)\mu(dx)$ is differentiable at every $u \in (a,b)$ and the derivative is

$$F'(u) = \int_\Omega \frac{\partial f}{\partial u}(x,u)\mu(dx).$$

Exercise 4. Show that,

$$\lim_{n\to\infty} \int_{(0,\infty)} \frac{\sin(t^n)}{t^n(1+t)} dt = \log 2.$$

Exercise 5. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f\colon \Omega \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be an absolutely integrable function.

(i) Prove that, the Fourier transform

$$\hat{f}(u) := \int e^{iux}\mu(dx), \quad u \in \mathbb{R}$$

is continuous on $\mathbb{R}$.

(ii) Suppose further that

$$\int |xf(x)|\mu(dx) < \infty.$$

Then the Fourier transform is differentiable on $\mathbb{R}$, and the derivative is

$$\hat{f}'(u) := i \int xe^{iux} f(x)\mu(dx).$$