Maß- und Integrationstheorie **Universität Mannheim**
HWS 2019 Dr. H. Pitters, Dr. Q. Shi

Sheet 10

For the exercise class 14.11.2019.

We denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel sigma-algebra on $\mathbb{R}^d$ (see *Beispiel 3.13*).

Exercise 1. (i) Recall *Theorem 4.5*.

(ii) Let $f\colon \mathbb{R} \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be defined by $f(x) = x^2$. Find the sigma-algebra $\sigma(f)$ and determine all measurable functions from $(\mathbb{R}, \sigma(f))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Exercise 2. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space with $\mu(\Omega) > 0$ and $f\colon \Omega \to (-\infty, +\infty)$ be a measurable function. Show that for every $\epsilon > 0$, there exists $A \in \mathcal{A}$ with $\mu(A) > 0$, such that for all $\omega, \omega' \in A$,

$$|f(\omega) - f(\omega')| < \epsilon.$$

Hint: For every $r \in \mathbb{Q}$, consider $A_r := \{\omega \in \Omega : f(\omega) \in (r - \epsilon/2, r + \epsilon/2)\}$.

Exercise 3. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space with $\mu(\Omega) < \infty$. For every $n \in \mathbb{N}$ let $f_n\colon \Omega \to [-\infty, +\infty]$ be a measurable function. Suppose that $f\colon \Omega \to [-\infty, +\infty]$ is a measurable function such that $\lim_{n\to\infty} f_n(\omega) = f(\omega)$ for μ-almost all $\omega \in \Omega$.

(i) For every $n, k \in \mathbb{N}$, define the set

$$E_{n,k} := \bigcup_{j \geq n} \left\{ \omega \in \Omega \ : \ |f_j(\omega) - f(\omega)| > \frac{1}{k} \right\}.$$

Show that $E_{n,k} \in \mathcal{A}$ and for every $\eta > 0$, there exists $N \in \mathbb{N}$ such that $\mu(E_{N,k}) < \eta$.

(ii) Prove that for every $\epsilon > 0$, there exists $A \in \mathcal{A}$ with $\mu(A) < \epsilon$, such that f_n converges uniformly to f on $\Omega \setminus A$.

(iii) Give a counter-example of the result of (ii) if we suppose that $\mu(\Omega) = \infty$.

Exercise 4. Consider a measure space $(\Omega, \mathcal{F}, \mu)$. Let $(f_n, n \geq 1)$ and f be measurable functions from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We say f_n converges to f in measure μ if for every $\epsilon > 0$,

$$\mu(|f - f_n| > \epsilon) \to 0.$$

(i) For every $k \geq 1$, let $E_k \in \mathcal{F}$. Suppose that $\sum_{k \geq 1} \mu(E_k) < \infty$. Show that $\mu(\limsup_{k \geq 1} E_k) = 0$.

(ii) If f_n converges to f in measure μ, then there exists a subsequence $(f_{n_k})_{k \geq 1}$, such that f_{n_k} converges to f μ-a.e. (μ-*f.ü.*).

(iii) Are monotone convergence theorem and dominated convergence theorem still true, if the assumption is convergence in measure?