Optimization in Machine Learning **Universität Mannheim**
HWS 2026 Prof. Leif Döring, Benedikt Wille

Sheet 4

For the exercise class on the 23.11.2026 at 12:00.

Exercises to be solved in the exercise class

Exercise 1 (Conditional Expectation)**.** **(0 Points)**

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $\mathcal{F}$ a subalgebra of $\mathcal{A}$ and X, Y random vectors. Prove for $\mathcal{F}$-measurable $X \in \mathbb{R}^d$ that we have

$$\mathbb{E}[\langle X, Y\rangle \,|\, \mathcal{F}] = \langle X, \mathbb{E}[Y \,|\, \mathcal{F}]\rangle$$

Exercise 2 (Conditional expectation)**.** **(0 Points)**

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an underlying probability space.

(i) Let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra and let $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Prove that

  (a) for any $\lambda \in \mathbb{R}$ there holds $\mathbb{E}[\lambda X + Y \,|\, \mathcal{G}] = \lambda \mathbb{E}[X \,|\, \mathcal{G}] + \mathbb{E}[Y \,|\, \mathcal{G}]$.

  (b) if $X \geq Y$ $\mathbb{P}$-almost surely, then $\mathbb{E}[X \,|\, \mathcal{G}] \geq \mathbb{E}[Y \,|\, \mathcal{G}]$ $\mathbb{P}$-a.s..

  (c) $|\mathbb{E}[X \,|\, \mathcal{G}]| \leq \mathbb{E}[|X| \,|\, \mathcal{G}]$.

(ii) Let Y_1,Y_2 be iid. random variables with

$$\mathbb{P}(Y_1 = 2) = \mathbb{P}(Y_1 = 0.5) = \frac{1}{2}\,,$$

and set

$$S_0 = 2, \quad S_k = S_0 \cdot \prod_{i=1}^{k} Y_i\,,\ k = 1, 2\,.$$

Compute $\mathbb{E}[S_2 \,|\, \mathcal{F}_1]$, where $\mathcal{F}_1 = \sigma(S_1)$.

(iii) Let X, Y be independent Bernoulli distributed random variables with parameter $p \in [0, 1]$ and define $Z := \mathbb{1}_{\{X+Y=0\}}$.

  (a) Compute $\mathbb{E}[X \,|\, \sigma(Z)]$ and $\mathbb{E}[Y \,|\, \sigma(Z)]$.

  (b) When are these random variables independent?

  Hint. *You may use the fact that a real valued random variable is independent from itself if and only if it is a constant.*

Exercises to be prepared at home for the next exercise class at 30.11.2026

Exercise 3 (Convexity and Expectation)**.** **(0 Points)**

Let Z be a random variable. Let $f(x) := f(x, Z)$ be a random function ($f(x, \omega) = f(x, Z(\omega))$ if you want) and its expectation

$$F(x) = \mathbb{E}[f(x)]$$

Is f almost surely convex if and only if F is convex? Prove or disprove both directions.

Exercise 4 (Martingales)**.** **(0 Points)**

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an underlying probability space.

(i) Let $Y_1, \dots, Y_N$ be iid. random variables with $\mathbb{E}[Y_1] = 1$ and $\mathbb{E}[|Y_1|] < \infty$, let $\mathcal{F}_k := \sigma(Y_1, \dots, Y_k)$ and define

$$S_0 = Y_0 := 1, \quad S_k = Y_1 \cdot \dots \cdot Y_k = \prod_{i=1}^{k} Y_i \,, \; k = 1, \dots, N \,.$$

Prove that $(S_k)_{k=1,\dots,N}$ is a Martingale with respect to the filtration $(\mathcal{F}_k)_{k=1,\dots,N}$.

(ii) Let $(X_k)_{k\in\mathbb{N}}$ and $(Y_k)_{k\in\mathbb{N}}$ be two martingales. Prove that $(aX_k + bX_k)_{k\in\mathbb{N}}$ is a martingale for any $a, b \in \mathbb{R}$.

(iii) Let $(X_k)_{k\in\mathbb{N}}$ and $(Y_k)_{k\in\mathbb{N}}$ be two super-martingales. Prove that $(\min\{X_k, Y_k\})_{k\in\mathbb{N}}$ is a super-martingale.