Lehrstuhl für Mathematik IV
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Calculus of Variations and Applications

- course outline -

1 Abstract

In this course we focus on minimization problems that involve integral functionals which are defined on scalar functions, which are in turn defined on an open bounded domain $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary. Thus, our goal is to minimize $I[u]$, where

$$I[u] := \int_\Omega L\big(\nabla u(x), u(x), x\big)\,\mathrm{d}x, \qquad u : \Omega \to \bar{\mathbb{R}},$$

under certain conditions on the boundary values of u, on $L : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and possibly under further constraints. The basic questions on these problems are existence, uniqueness and regularity of minimizers, and the aim of the course is to be exposed to the basic theory underlying these questions. Under precise assumptions on the function $L = L(\xi, u, x)$, conditions on existence and uniqueness are presented. For instructive reasons, the difficult question of regularity of minimizers is detailed only for $L = \frac{1}{p}|\xi|^p$, $p > 1$. Side consequences of our exposition will be the existence of area minimizing sets as well as the Euclidean isoperimetric inequality.

2 Bibliography

(D) DACOROGNA, B. **Introduction to the Calculus of Variations**. 3rd edition. ICP 2015,

(Ev) EVANS, L. C. **Partial Differential Equations**. 2nd edition. AMS 2010,

(MZ) MALÝ, J.; ZIEMER, W. P. **Fine Regularity of Solutions of Elliptic PDE**. AMS 1997.

Our basic references are (D) and the 8th chapter of (Ev). For the theory of Sobolev spaces we follow Chap. 5 in (Ev) as well as §1.2-§1.3 of (MZ). For regularity theory we follow §2.3 of (MZ).

3 Calendar

LECTURE #01 - Monday, 02/09 (10:15-11:45) Introduction to the Calculus of Variations with the aid of some examples from geometry, physics and economics where one is led to minimize integral functionals, in some cases under further constraints. In particular, Dido's isoperimetric problem, the brachistochrone problem, electrostatics, stationary states in quantum mechanics and optimal savings and consumption.

LECTURE #02 - Wednesday, 04/09 (08:30-10:00) Lebesgue-measure/measurable functions/ integral. Summable functions. Basic theorems of analysis (Fatou's lemma, monotone convergence theorem, dominated convergence theorem, absolute continuity of the integral, density of $C_c(\mathbb{R}^n)$ in the space of summable functions). For all these, consult the relevant file on the webpage of the course.

We proved:

Theorem 3.1. *If $f, \{f_k\}_{k\in\mathbb{N}}$ are summable in $\mathbb{R}^n$ and $\int_{\mathbb{R}^n} |f_k - f| \mathrm{d}\mathscr{L}^n \to 0$ as $k \to \infty$, then there is a subsequence $\{f_{k_j}\}_{j\in\mathbb{N}}$ such that $f_{k_j} \to f$ a.e. in $\mathbb{R}^n$.*

I. Basic theory of L^p Spaces

LECTURE #03 - Monday, 09/09 (10:15-11:45) L^p spaces and their properties:

$\triangleq$ *Essential supremum*: For any measurable $g : \mathbb{R}^n \to \bar{\mathbb{R}}$ we set

$$\text{ess sup}_{\mathbb{R}^n}\, g := \begin{cases} \infty & \text{if } \mathscr{L}^n\big(\{x \in \mathbb{R}^n \mid g(x) > \alpha\}\big) > 0 \ \ \forall \alpha \in \mathbb{R}, \\ \inf\{\alpha \in \mathbb{R} \mid \mathscr{L}^n\big(\{x \in \mathbb{R}^n \mid g(x) > \alpha\}\big) = 0\} & \text{otherwise.} \end{cases}$$

$\triangleq$ *The space L^p, $p \in [1,\infty]$*:

$$L^p \equiv L^p(\mathbb{R}^n) := \{\text{all measurable functions } f : \mathbb{R}^n \to \bar{\mathbb{R}} \text{ such that } |f|^p \text{ is summable}\},$$

$$L^\infty \equiv L^\infty(\mathbb{R}^n) := \{\text{all measurable functions } f : \mathbb{R}^n \to \bar{\mathbb{R}} \text{ such that ess sup} |f| \text{ is finite}\}.$$

We proved:

Theorem 3.2 (`Hölder's inequality`)**.** *Let $p, q \in [1,\infty]$ satisfying $1/p + 1/q = 1$. If $f \in L^p$, $g \in L^q$ then*

$$\int_{\mathbb{R}^n} |fg| \, \mathrm{d}\mu \leq \begin{cases} \big(\text{ess sup}_{\mathbb{R}^n} |g|\big) \int_{\mathbb{R}^n} |f| \, \mathrm{d}\mathscr{L}^n & \text{if } p = 1, \\ \Big(\int_{\mathbb{R}^n} |f|^p \, \mathrm{d}\mathscr{L}^n\Big)^{1/p} \Big(\int_{\mathbb{R}^n} |g|^q \, \mathrm{d}\mathscr{L}^n\Big)^{1/q} & \text{if } 1 < p < \infty, \\ \big(\text{ess sup}_{\mathbb{R}^n} |f|\big) \int_{\mathbb{R}^n} |g| \, \mathrm{d}\mathscr{L}^n & \text{if } p = \infty. \end{cases}$$

We proved:

Theorem 3.3 (`Minkowski's inequality`)**.** *If $f, g \in L^p$ with $p \in [1,\infty)$, then*

$$\left(\int_{\mathbb{R}^n} |f+g|^p \, \mathrm{d}\mathscr{L}^n\right)^{1/p} \leq \left(\int_{\mathbb{R}^n} |f|^p \, \mathrm{d}\mathscr{L}^n\right)^{1/p} + \left(\int_{\mathbb{R}^n} |g|^p \, \mathrm{d}\mathscr{L}^n\right)^{1/p}.$$

If $f, g \in L^\infty$ then $\text{ess sup}_{\mathbb{R}^n} |f+g| \leq \text{ess sup}_{\mathbb{R}^n} |f| + \text{ess sup}_{\mathbb{R}^n} |g|$.

Hence, if $p \in [1,\infty]$, the function $\|\cdot\|_p : L^p \to [0,\infty)$ given by

$$\|f\|_p \equiv \|f\|_{L^p} \equiv \|f\|_{L^p(\mathbb{R}^n)} := \begin{cases} \Big(\int_{\mathbb{R}^n} |f|^p \, \mathrm{d}\mathscr{L}^n\Big)^{1/p} & \text{if } p \in [1,\infty) \\ \text{ess sup}_{\mathbb{R}^n} |f| & \text{if } p = \infty, \end{cases},$$

defines a norm on the linear space L^p. From Theorem 3.1 we readily deduce that convergence in the norm of L^p for any $p \in [1,\infty)$, implies convergence almost everywhere up to a subsequence. More is true:

Theorem 3.4 (L^p `is a Banach space`)**.** *Let $1 \leq p \leq \infty$ and suppose $f_k : \mathbb{R}^n \to \bar{\mathbb{R}}$, $k \in \mathbb{N}$, is a Cauchy sequence in L^p. There exists then a subsequence $\{f_{k_l}\}_{l\in\mathbb{N}}$ such that*

(i) $|f_{k_l}| \leq F$ a.e. in $\mathbb{R}^n$, for any $k \in \mathbb{N}$, and some nonnegative $F \in L^p$,

(ii) $f_{k_l} \to f$ a.e. in $\mathbb{R}^n$, and some $f : \mathbb{R}^n \to \bar{\mathbb{R}}$.

Given this, we proved further: (I) by applying Fatou's lemma to the sequence $g_l := |f_{k_l}|^p$, that $f \in L^p$, (II) by applying then the dominated convergence theorem for the sequence $h_l := |f_{k_l} - f|^p$, that $\lim_{k\to\infty} \|f_{k_l} - f\|_p \to 0$, *(III) that* $\lim_{k\to\infty} \|f_k - f\|_p \to 0$.

$\triangleq$ *Continuous linear functionals of L^p*: A *linear functional of L^p* is a map $\ell : L^p \to \mathbb{R}$ for which

$$\ell(\alpha f + \beta g) = \alpha \ell(f) + \beta \ell(g) \quad \forall\, f, g \in L^p,\ \forall\, \alpha, \beta \in \mathbb{R}.$$

Such a functional ℓ is called *continuous* if

$$\lim_{k\to\infty} \ell(f_k) \to 0 \quad \text{whenever } f_k \to 0 \text{ in } L^p,$$

and *bounded* if

$$|\ell(f)| \le K \|f\|_p \quad \forall f \in L^p.$$

Proposition 3.5. *A linear functional ℓ of L^p is continuous if and only if*

(i) $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ *such that* $|\ell(f)| \le \varepsilon$ *whenever* $\|f\|_p \le \delta$,

(ii) it is bounded.

$\triangleq$ *Dual space*: The set of all continuous linear functionals of L^p is called the *dual* of L^p and is denoted by $\left(L^p\right)^*$. It is a normed linear space with the norm

$$\|\ell\| = \sup\{|\ell(f)| \mid \|f\|_p \le 1\}.$$

Theorem 3.6 (`the dual of` L^p)**.** *(i) If $p \in [1, \infty)$ then $(L^p)^* = L^q$, where $1/p + 1/q = 1$, in the sense that*

$$\forall\, \ell \in (L^p)^*,\ \exists!\, v_\ell \in L^q \text{ such that } \ell(g) = \int_{\mathbb{R}^n} v_\ell g \, \mathrm{d}\mathscr{L}^n \quad \forall\, g \in L^p.$$

(ii) If $p \in [1, \infty]$, a functional $\ell : L^p \to \mathbb{R}$ defined for all $g \in L^p$ by

$$\ell(g) = \int_{\mathbb{R}^n} vg \, \mathrm{d}\mathscr{L}^n \quad \text{for some } v \in L^q,\ \text{ where } 1/p + 1/q = 1,$$

is always a member of $(L^p)^*$ and moreover $\|\ell\| = \|v\|_q$.

$\triangleq$ *Weak convergence in L^p*: Let $1 \le p \le \infty$. A sequence $\{f_k \in L^p\}_{k\in\mathbb{N}}$ is said to *converge weakly in L^p* to $f \in L^p$, denoted by $f_k \rightharpoonup f$ in L^p, whenever

$$\ell(f_k) \to \ell(f) \qquad \text{for all } \ell \in (L^p)^*.$$

Observe that for $1 \le p < \infty$, Theorem 3.6 implies $f_k \rightharpoonup f$ in L^p is equivalent to

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} (f_k - f) g \, \mathrm{d}\mathscr{L}^n = 0 \qquad \text{for all } g \in L^q,\ \text{ where } 1/p + 1/q = 1.$$

A sequence $\{f_k \in L^\infty\}_{k\in\mathbb{N}}$ is said to *converge weak-star in L^∞* to $f \in L^\infty$, denoted by $f_k \overset{\star}{\rightharpoonup} f$ in L^∞, whenever

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} (f_k - f) g \, \mathrm{d}\mathscr{L}^n = 0 \qquad \text{for all } g \in L^1.$$

Proposition 3.7. *Let $f \in L^p$, $p \in [1, \infty]$. Then*

$$l(f) = 0 \quad \forall\, l \in (L^p)^* \implies f = 0 \text{ a.e. in } \mathbb{R}^n.$$

Corollary 3.8. *Let $p \in [1, \infty]$ and $f_k \rightharpoonup g$ in L^p, $f_k \rightharpoonup h$ in L^p. Then $g = h$ a.e. in $\mathbb{R}^n$.*

Theorem 3.9 (`lower semi-continuity of norms`). *Suppose that $f_k \rightharpoonup g$ in L^p, $p \in [1,\infty]$. Then*

(i) if $p \in [1,\infty]$, then $\liminf_{k\to\infty} \|f_k\|_p \geq \|f\|_p$,

(ii) if $p \in (1,\infty)$ and $\lim_{k\to\infty} \|f_k\|_p = \|f\|_p$, then $f_k \to f$ in L^p.

Theorem 3.10 (`uniform boundedness principle`). *Let $p \in [1,\infty]$ and suppose $\{f_k \in L^p\}_{k\in\mathbb{N}}$ is such that $\{l(f_k)\}_{k\in\mathbb{N}}$ is bounded for any $l \in (L^p)^*$. Then $\{\|f_k\|_p\}_{k\in\mathbb{N}}$ is also bounded.*

Theorem 3.11 (`separability of` L^p). *L^p for $1 \leq p < \infty$ is separable; that is, it contains a dense countable set (this fails for $p = \infty$).*

It follows from the uniform boundedness principle (or from Theorem 3.9) that if $f_k \rightharpoonup f$ in L^p, where $p \in [1,\infty]$, then $\{\|f_k\|_p\}_{k\in\mathbb{N}}$ is bounded. The converse of this statement is true, modulo passing to a subsequence:

Theorem 3.12 (`Banach-Alaoglu theorem, or weak compactness property of` L^p). *Let $p \in (1,\infty]$. If $\{f_k \in L^p\}_{k\in\mathbb{N}}$ is such that $\{\|f_k\|_p\}_{k\in\mathbb{N}}$ is bounded, then there exists $f \in L^p$ and a subsequence $\{f_{k_l}\}_{l\in\mathbb{N}}$ such that $f_{k_l} \rightharpoonup f$ in L^p.*

LECTURE #04 - Wednesday, 11/09 (08:30-10:00) Approximation of L^p functions by smooth functions - mollification technique:

$\triangleq$ For any $\varepsilon > 0$ we give the following definitions:

A) The *standard mollifier* $\eta_\varepsilon : \mathbb{R}^n \to [0,\infty)$ given by $\eta_\varepsilon(x) := \varepsilon^{-n}\eta(x/\varepsilon)$, $x \in \mathbb{R}^n$, where the function $\eta : \mathbb{R}^n \to [0,\infty)$ is

$$\eta(x) := \begin{cases} c\exp\{(|x|^2-1)^{-1}\} & |x| < 1, \\ 0 & |x| \geq 1, \end{cases} \tag{1}$$

with the constant $c > 0$ being such that $\int_{\mathbb{R}^n} \eta(x)\,\mathrm{d}x = 1$.

B) The set $U_\varepsilon := \{x \in U \mid \operatorname{dist}(x,\partial U) > \varepsilon\}$, where U is any open subset of $\mathbb{R}^n$. If $U = \mathbb{R}^n$ then $U_\varepsilon = \mathbb{R}^n$.

C) Given $u \in L^1_{\mathrm{loc}}(U)$, where U is an open subset of $\mathbb{R}^n$, we define the *mollification of* u as the function $u_\varepsilon : U_\varepsilon \to \mathbb{R}$ given by $u_\varepsilon := \eta_\varepsilon * u$, that is

$$u_\varepsilon(x) := \int_U \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y, \qquad x \in U_\varepsilon.$$

We proved:

Theorem 3.13 (`properties of mollifiers - part I`). *Let $U \subseteq \mathbb{R}^n$ be open.*

(i) If $u \in L^1_{\mathrm{loc}}(U)$, then for all $\varepsilon > 0$ we have $u_\varepsilon \in C^\infty(U)$.

(ii) If $u \in C(U)$, then $u_\varepsilon \to u$ locally uniformly in U.

(iii) If $u \in L^p_{\mathrm{loc}}(U)$ for some $p \in [1,\infty)$, then $u_\varepsilon \to u$ in $L^p_{\mathrm{loc}}(U)$.

Next we approximated the characteristic function of a set as follows:

If K is a compact subset of the open set $U \subseteq \mathbb{R}^n$, then there exists $\varepsilon > 0$ such that $K_+ := \{x \in \mathbb{R}^n \mid \operatorname{dist}(x,K) \le \varepsilon\}$ is again a compact subset of U with $K \subset K_+^o$ (because the distance of a compact set to a disjoint closed set is always positive). More generally, we write $A \Subset B$ to declare that $A,\ B$ are open subsets of $\mathbb{R}^n$ such that $\bar{A} \subset B$ and $\bar{A}$ is bounded (and thus compact). Hence, the above fact says that given $V \Subset U$, there exists W such that $V \Subset W \Subset U$.

$\triangleq$ The support of a continuous function $g : \mathbb{R}^n \to \mathbb{R}$, denoted by $\operatorname{supp}\{g\}$, is defined by

$$\operatorname{supp}\{g\} := \overline{\{x \in \mathbb{R}^n \mid g(x) \neq 0\}}.$$

We proved

Lemma 3.14 (`elementary form of Urysohn's lemma`)**.** *Let $U \subseteq \mathbb{R}^n$ be open and $K \subset U$ be compact. There exists a function $g : \mathbb{R}^n \to \mathbb{R}$ such that*

(i) $g \in C_c(\mathbb{R}^n)$*; that is, g is continuous with compact support.*

(ii) $0 \le g(x) \le 1$ *for all* $x \in \mathbb{R}^n$.

(iii) $g(x) = 1$ *for all* $x \in K$.

(iv) $\operatorname{supp}\{g\} \subset U$.

TUTORIAL #01 - Thursday, 12/09 (15:30-17:00) Solutions to exercises - Assignment #01. We have generalised the last exercise to the so called fundamental lemma of the Calculus of Variations

Lemma 3.15 (Fundamental lemma of the Calculus of Variations)**.** *(i) If $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is such that $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n} f\eta dx \ge 0 \quad \forall\, \eta \in C_c^\infty(\mathbb{R}^n),\ \eta \ge 0 \qquad \Big(\text{or, } \int_{\mathbb{R}^n} f\eta dx = 0 \quad \forall\, \eta \in C_c^\infty(\mathbb{R}^n)\Big), \tag{2}$$

then $f \ge 0$ a.e. in $\mathbb{R}^n$ (or, $f = 0$ a.e. in $\mathbb{R}^n$). As an application, we showed that

(ii) if $f, g : \mathbb{R}^n \to \mathbb{R}$ are such that $f, g \in L^1_{\text{loc}}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} f\eta dx = 0 \quad \forall\, \eta \in C_c^\infty(\mathbb{R}^n) \text{ with } \int_{\mathbb{R}^n} g\eta dx = 0,$$

then there exists $\lambda \in \mathbb{R}$ such that $f = \lambda g$ for a.e. $x \in \mathbb{R}^n$. Observe this implies that if $\mathscr{L}^n(\Omega) < \infty$ and $f : \Omega \to \mathbb{R}$ is such that $f \in L^1_{\text{loc}}(\Omega)$ and

$$\int_\Omega f\eta dx = 0 \quad \forall\, \eta \in C_c^\infty(\Omega) \text{ with } \int_\Omega \eta dx = 0,$$

then f coincides a.e. in Ω with a constant function.

II. Some one dimensional classical problems in the Caclulus of Variations

LECTURE #05 - Monday, 16/09 (10:15-11:45) One dimensional classical problems - Part I (Critical points, convexity, uniqueness. Other forms of the Euler-Lagrange equation. Examples).

LECTURE #06 - Wednesday, 18/09 (08:30-10:00) One dimensional classical problems - Part II (Further examples, the Poincaré inequality, Mania example).

TUTORIAL #02 Thursday, 19/09 (15:30-17:00) Solutions to exercises - Assignment #02. We proved a slightly more general result than the second part of the second exercise, known as the Brezis-Lieb lemma.

III. Basic theory of Sobolev Spaces

LECTURE #07 - Monday, 23/09 (15:30-17:00) We have introduced the notion of weak derivative and then defined the Sobolev space $W^{k,p}(U)$, where $U \subseteq \mathbb{R}^n$ is open, $p \in [1,\infty]$ and $k \in \mathbb{N} \cup \{0\}$. We discussed basic properties this space enjoys. We proved it is a Banach space and also the following local approximation theorem

Theorem 3.16 (`properties of mollifiers - part II`). *Let $U \subseteq \mathbb{R}^n$ be open. Assume $u \in W^{k,p}_{\mathrm{loc}}(U)$ for some $p \in [1,\infty)$ and $k \in \mathbb{N}$. Define u_ε the same way as in Theorem 3.13. Then $u_\varepsilon \to u$ in $W^{k,p}_{\mathrm{loc}}(U)$.*

LECTURE #08 - Thursday, 26/09 (13:45-15:15) We proved the following theorems:

Theorem 3.17 (`global approximation by smooth functions`[1]). *Let $U \subset \mathbb{R}^n$ be open and bounded. Assume $u \in W^{k,p}(U)$ for some $p \in [1,\infty)$ and $k \in \mathbb{N}$. Then there exists a sequence $\{u_k \in C^\infty(U) \cap W^{k,p}(U)\}_{k \in \mathbb{N}}$ such that $u_k \to u$ in $W^{k,p}(U)$.*

Theorem 3.18 (`global approximation by functions smooth up to` ∂U). *Let U be a bounded open subset of $\mathbb{R}^n$, with boundary of class $\mathscr{C}^1$. Assume $u \in W^{k,p}(U)$ for some $p \in [1,\infty)$ and $k \in \mathbb{N}$. Then there exists a sequence $\{u_k \in C^\infty(\bar{U})\}_{k \in \mathbb{N}}$ such that $u_k \to u$ in $W^{k,p}(U)$.*

TUTORIAL #03 Thursday, 26/09 (15:30-17:00) We proved the following theorem:

Theorem 3.19 (`extension of Sobolev functions`). *Let $U \subset \mathbb{R}^n$ be open, bounded with boundary of class $\mathscr{C}^1$. Select an open set V such that $U \Subset V$. Then there exists a bounded linear operator $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$, $p \in [1,\infty)$, such that for any $u \in W^{1,p}(U)$:*

(i) $Eu = u$ *a.e. in* U,

(ii) $\mathrm{support}(Eu) \subset V$, *and*

(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(U)}$, *where the constant C is independent of u.*

Solutions to exercises 1 & 3 - Assignment #03.

LECTURE #09 - Monday, 30/09 (15:30-17:00) We proved the following theorem:

Theorem 3.20 (`trace of Sobolev functions`). *Let $U \subset \mathbb{R}^n$ be open, bounded with boundary of class $\mathscr{C}^1$. Then there exists a bounded linear operator $T : W^{1,p}(U) \to L^p(\partial U)$, $p \in [1,\infty)$, such that:*

(i) $Tu = u|_{\partial U}$ *for any* $u \in W^{1,p}(U) \cap C(\bar{U})$, *and*

(ii) $\|Tu\|_{L^p(\partial U)} \le C\|u\|_{W^{1,p}(U)}$ *for any* $u \in W^{1,p}(U)$, *where the constant C is independent of u.*

Corollary 3.21. *Let $U \subset \mathbb{R}^n$ be open, bounded with boundary of class $\mathscr{C}^1$. Then for any $u \in W^{1,p}_0(U)$, $p \in [1,\infty)$, we have $Tu = 0$ on ∂U.*[2]

Solution to exercise 2 - Assignment #03.

LECTURE #10 - Monday, 07/10 (15:30-17:00) We proved the following lemmata:

[1] this theorem is true also in case U is unbounded

[2] the converse assertion is also true; that is: *Under the same assumptions on U as in Corollary 3.21, if $u \in W^{1,p}(U)$, $p \in [1,\infty)$, has $Tu = 0$ on ∂U, then $u \in W^{1,p}_0(U)$.*

Lemma 3.22. *Let $u \in C_c^1(U)$, where $U \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, is open. Then we have the representation formula*

$$u(x) = \frac{1}{n\omega_n} \int_U \frac{(x-z) \cdot \nabla u(z)}{|x-z|^n} \mathrm{d}z \qquad \forall\, x \in U,$$

where $\omega_n = \mathscr{L}^n(B_1)$.

It readily follows from this lemma that

$$|u(x)| \leq \frac{1}{n\omega_n} \int_U \frac{|\nabla u(z)|}{|x-z|^{n-1}} \mathrm{d}z \qquad \forall\, x \in U. \tag{3}$$

Lemma 3.23. *Let $U \subset \mathbb{R}^n$, $n \in \mathbb{N} \setminus \{1\}$, be open with $\mathscr{L}^n(U) < \infty$. Then for any $\alpha \in [0,n)$ we have*

$$\int_U \frac{1}{|x-z|^\alpha} \mathrm{d}z \leq \int_{B_{R_U}(x)} \frac{1}{|x-z|^\alpha} \mathrm{d}z \qquad \forall\, x \in U,$$

where R_U is such that $\mathscr{L}^n(U) = \mathscr{L}^n(B_{R_U})$; that is $R_U := \big(\mathscr{L}^n(U)/\omega_n\big)^{1/n}$.

Using polar coordinates we easily computed the exact value of the right hand side in the above inequality. We found

$$\int_U \frac{1}{|x-z|^\alpha} \mathrm{d}z \leq \frac{n\omega_n^{\alpha/n}}{n-\alpha} \big(\mathscr{L}^n(U)\big)^{1-\alpha/n} \qquad \forall\, x \in U. \tag{4}$$

Applying Holder's inequality to (3) and then using (4) with $\alpha = (n-1)p/(p-1)$ we proved the following proposition

Proposition 3.24. *Let $U \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be open with $\mathscr{L}^n(U) < \infty$. Let also $p > n$. Then for any $u \in C_c^1(U)$ we have the estimate*

$$\|u\|_{L^\infty(U)} \leq n^{-1/p} \omega_n^{-1/n} \Big(\frac{p-1}{p-n}\Big)^{1-1/p} \big(\mathscr{L}^n(U)\big)^{1/n-1/p} \|\nabla u\|_{L^p(U)}.$$

We proved the following lemma:

Lemma 3.25. *Let $u \in C^1(U)$, where $U \subseteq \mathbb{R}^n$, $n \in \mathbb{N} \setminus \{1\}$, is open. Let $B_{2r}(x_0) \subset U$. Then we have the following local version of inequality* (3)

$$|u(x) - u_{B_r(x_0)}| \leq \frac{2^n}{n\omega_n} \int_{B_r(x_0)} \frac{|\nabla u(z)|}{|x-z|^{n-1}} \mathrm{d}z \qquad \forall\, x \in B_r(x_0),$$

where $u_{B_r(x_0)} := \frac{1}{\mathscr{L}^n(B_r(x_0))} \int_{B_r(x_0)} u(y)\mathrm{d}y$.

LECTURE #11 - Thursday, 10/10 (13:45-15:15) We proved

Proposition 3.26. *Let $U \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, be open. Then for any $u \in C_c^1(U)$ we have the estimate*

$$\sup_{\substack{x,y \in U \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^{1-n/p}} \leq C(n,p) \|\nabla u\|_{L^p(U)}.$$

Given any $\alpha \in (0,1]$ and open $U \subseteq \mathbb{R}^n$, we defined the α^{th} Hölder seminorm $[u]_{C^{0,\alpha}(\bar{U})}$ and norm $\|u\|_{C^{0,\alpha}(\bar{U})}$ of a given function $u : U \to \mathbb{R}$. Then we defined the Hölder space $C^{k,\alpha}(\bar{U})$, $k \in \mathbb{N} \cup \{0\}$, and stated that this is a Banach space.

Using Proposition 3.24 we proved

Theorem 3.27. *Let $U \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be open with $\mathscr{L}^n(U) < \infty$. If $p > n$ then $u \in W_0^{1,p}(U) \subset L^\infty(U)$ with the estimate*

$$\|u\|_{L^\infty(U)} \leq C(n,p)\big(\mathscr{L}^n(U)\big)^{1/n-1/p}\|\nabla u\|_{L^p(U)}.$$

Using Proposition 3.26 we proved

Theorem 3.28. *Let $U \subset \mathbb{R}^n$, $n \in \mathbb{N}\setminus\{1\}$, be open, bounded and with boundary of class $\mathscr{C}^1$. If $p > n$ then any function $u \in W^{1,p}(U)$ has a version[3] $u^\star \in C^{0,1-n/p}(\bar{U})$ with the estimate*

$$\|u^\star\|_{C^{0,1-n/p}(\bar{U})} \leq C(n,p,U)\|\nabla u\|_{W^{1,p}(U)}.$$

From now on, if the assumptions of the above theorem are satisfied, we will readily replace a given $u \in W^{1,p}(U)$ by its continuous version.

TUTORIAL #04 Thursday, 10/10 (15:30-17:00) Let $U \subseteq \mathbb{R}^n$, $n \in \mathbb{N}\setminus\{1\}$, be open. Using Hölder's inequality we proved that if we know

$$\|u\|_{L^{n/(n-1)}(U)} \leq C(n)\|\nabla u\|_{L^1(U)} \qquad \forall\, u \in W_0^{1,1}(U), \tag{5}$$

then we know also for any $p \in (1,n)$ that

$$\|u\|_{L^{np/(n-p)}(U)} \leq C(n,p)\|\nabla u\|_{L^p(U)} \qquad \forall\, u \in C_c^\infty(U).$$

Solutions to exercises - Assignment #04.

LECTURE #12 - Monday, 14/10 (15:30-17:00) We proved (5) for functions $u \in C_c^1(\mathbb{R}^n)$. Its extension to $W_0^{1,1}(U)$ for any open $U \subseteq \mathbb{R}^n$ is done as usual. We summarize these in the following theorem

Theorem 3.29. *Let $U \subset \mathbb{R}^n$, $n \in \mathbb{N}\setminus\{1\}$, be open and let $p \in [1,n)$. Then*

$$\|u\|_{L^{np/(n-p)}(U)} \leq C(n,p)\|\nabla u\|_{L^p(U)} \qquad \forall\, u \in W_0^{1,p}(U).$$

Combining the interpolation theorem in L^p spaces with the above result, we proved a multiplicative Sobolev inequality. Also, from (3), using Holder's inequality with three exponents we proved

Theorem 3.30. *Let $U \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be open with $\mathscr{L}^n(U) < \infty$. Then we have $W_0^{1,n}(U) \subset L^q(U)$ for any $q \in [1,\infty)$.*

LECTURE #13 - Thursday, 17/10 (13:45-15:15) We proved the first part of the following theorem

Theorem 3.31 (`Rellich-Kondrachov theorem, or compactness property of` $W^{1,p}$). *Let $U \subset \mathbb{R}^n$ be open, bounded with boundary of class $\mathscr{C}^1$. Let $p \geq 1$. If $\{u_k \in W^{1,p}(U)\}_{k\in\mathbb{N}}$ is such that $\{\|u_k\|_{W^{1,p}(U)}\}_{k\in\mathbb{N}}$ is bounded, then*

(i) in case $p \in [1,n)$ there exists $u \in L^q(U)$ and a subsequence $\{u_{k_l}\}_{l\in\mathbb{N}}$ such that $u_{k_l} \to u$ in $L^q(U)$ for any $q \in \big[1, np/(n-p)\big)$.

(ii) in case $p = n$ there exists $u \in L^q(U)$ and a subsequence $\{u_{k_l}\}_{l\in\mathbb{N}}$ such that $u_{k_l} \to u$ in $L^q(U)$ for any $q \in [1,\infty)$.

(iii) in case $p > n$ there exists $u \in C^{0,\alpha}(\bar{U})$ and a subsequence $\{u_{k_l}\}_{l\in\mathbb{N}}$ such that $u_{k_l} \to u$ in $C^{0,\alpha}(\bar{U})$ for any $\alpha \in [0, 1-n/p)$.

[3] this means $u = u^\star$ a.e.in U

TUTORIAL #05 Thursday, 17/10 (15:30-17:00) Solutions to exercises 2 and 3 - Assignment #05. We also solved the following exercise:

Let $1 \le p < n$, $n \in \mathbb{N} \setminus \{1\}$, U be an open subset of $\mathbb{R}^n$. Consider a function $u \in W^{1,p}(U)$ such that

$$\mathscr{L}^n\big(\{x \in U \mid u(x) = 0\}\big) \ge \alpha \qquad \textit{for some } \alpha \in \big(0, \mathscr{L}^n(U)\big].$$

Prove there exists a constant C, independent on u, such that

$$\|u\|_{L^p(U)} \le C\|\nabla u\|_{L^p(U)}.$$

Finally, we proved the fact that *translation is continuous in the L^p-norm*; that is

$$f \in L^p(\mathbb{R}^n) \Longrightarrow \lim_{h \to 0} \int_{\mathbb{R}^n} |f(x+h) - f(x)|^p \mathrm{d}\mathscr{L}^n(x) = 0.$$

This property of L^p spaces was used in the proof of Theorem 3.18.

IV. EXISTENCE OF MINIMIZERS
(The Direct Method in the Calculus of Variations)

LECTURE #14 - Monday, 21/10 (15:30-17:00) First we have recalled the proof (from Analysis I or II) of the attainability of the minimum for a real valued continuous function defined on a closed and bounded subset of $\mathbb{R}^n$. Then we tried to follow the same line of proof to show attainability of the minimum for a real valued strongly continuous functional $\mathscr{F}$ defined on a closed and bounded subset of $L^2(U)$. Since such a subset is only weakly compact (by Theorem 3.12), we see that the strong continuity assumption on $\mathscr{F}$ is useless. Instead, one is naturally led to require $\mathscr{F}$ to be *weakly lower semicontinuous*; that is, $\liminf_{k\to\infty} \mathscr{F}(\psi_j) \ge \mathscr{F}(\psi)$, whenever $\psi_j \rightharpoonup \psi$ in $L^2(U)$.

Following the above ideas we proved the existence of a minimizer for a model problem, the so called *Dirichlet problem*:

Theorem 3.32. *Let $U \subset \mathbb{R}^n$ be open, bounded with boundary of class $\mathscr{C}^1$ and suppose $g \in W^{1,2}(U)$. Then the problem*

$$\inf_{u \in \mathscr{A}_g} \left\{ I[u] := \int_U |\nabla u|^2 \, \mathrm{d}\mathscr{L}^n \right\}, \textit{ where } \mathscr{A}_g := \big\{u \in W^{1,2}(U) \mid u = g \textit{ on } \partial U, \textit{ in the trace sense}\big\},$$

admits a solution $\bar{u} \in \mathscr{A}_g$.

LECTURE #15 - Thursday, 24/10 (13:45-15:15) Let $U \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, be open and fix $q \in (1, \infty)$.

$\triangleq$ A functional $L : W^{1,q}(U) \to \mathbb{R}$ is called *weakly lower semicontinuous* provided

$$\liminf_{k\to\infty} I[u_k] \ge I[u], \qquad \text{whenever } u_k \rightharpoonup u \text{ in } W^{1,q}(U).$$

We proved:

Theorem 3.33. *Assume $L : \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$ is smooth (C^1 will do), bounded from below and in addition $L(\cdot, z, x)$ is convex for each $z \in \mathbb{R}$, $x \in U$, where U is further bounded and of class $\mathscr{C}^1$. Then the functional $I : W^{1,q}(U) \to \mathbb{R}$ defined by*

$$I[w] := \int_U L\big(\nabla w(x), w(x), x\big) \mathrm{d}\mathscr{L}^n(x), \qquad w \in W^{1,q}(U), \tag{6}$$

is weakly lower semicontinuous.

We proved:

Theorem 3.34 (`existence theorem`). *Additionally to the assumptions of the above theorem consider that*

$$L(\xi,z,x) \geq \alpha|\xi|^q - \beta \qquad \forall\, \xi \in \mathbb{R}^n,\ z \in \mathbb{R},\ x \in U,$$

for some constants $\alpha > 0$ and $\beta \geq 0$. Let $g \in W^{1,q}(U)$ and define the set

$$\mathscr{A}_g := \left\{ u \in W^{1,q}(U) \mid u = g \text{ on } \partial U, \text{ in the trace sense} \right\}. \tag{7}$$

Then there exists at least one $u \in \mathscr{A}_g$ such that $I[u] = \inf_{w\in\mathscr{A}_g} I[w]$.

TUTORIAL #06 Thursday, 17/10 (15:30-17:00) Solutions to exercises - Assignment #06.

V. Properties of minimizers

LECTURE #16 - Monday, 28/10 (15:30-17:00) Suppose $I[\cdot]$ is given by (6) with L being smooth (C^2 will do). Let $u \in \mathscr{A}_g \cap C^2(U)$ be such that $I[u] = \inf_{w\in\mathscr{A}_g} I[w]$.

- Computing the first variation of $I[\cdot]$, we showed that u has to be a solution of the following second order, quasilinear partial differential equation, in divergence form:

$$-\mathrm{div}_x\left\{\nabla_\xi L\big(\nabla u(x), u(x), x\big)\right\} + L_z\big(\nabla u(x), u(x), x\big) = 0 \qquad \text{in } U. \tag{8}$$

  This is the `Euler-Lagrange equation` associated with the functional given by (6).

- Computing the second variation of $I[\cdot]$, we showed that the following first order inequality has to be satisfied

$$\sum_{i,j=1}^{n} L_{\xi_i\xi_j}\big(\nabla u(x), u(x), x\big)\eta_i\eta_j \geq 0 \qquad \forall\, \eta \in \mathbb{R}^n,\ x \in U. \tag{9}$$

  This is a sort of convexity condition for L in the p-variable, since (exercise) whenever a function $f \in C^2(\mathbb{R}^n)$ is convex then the Hessian matrix $D^2 f$ has to be a nonnegative definite symmetric matrix on $\mathbb{R}^n$, hence in particular $\big(D^2 f(x)\eta\big)\cdot\eta \geq 0$ for any $\eta \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, or what is the same,

$$\sum_{i,j=1}^{n} f_{x_ix_j}(x)\eta_i\eta_j \geq 0 \qquad \forall\, \eta \in \mathbb{R}^n,\ x \in \mathbb{R}^n.$$

VI. Uniqueness of Minimizers

- Assuming $L = L(\xi,x)$ and that L is uniformly convex in the ξ-variable; that is, with some constant $\theta > 0$ there holds

$$\sum_{i,j=1}^{n} L_{\xi_i\xi_j}(\xi,x)\eta_i\eta_j \geq \theta|\eta|^2 \qquad \forall\, \xi,\eta \in \mathbb{R}^n,\ x \in U,$$

  we proved uniqueness of $u \in \mathscr{A}_g$ such that $I[u] = \inf_{w\in\mathscr{A}_g} I[w]$.

- Assuming $L = L(\xi,z,x)$ and that L is strictly convex in the (ξ,z)-variables, we proved uniqueness of $u \in \mathscr{A}_g$ such that $I[u] = \inf_{w\in\mathscr{A}_g} I[w]$.

VII. THE PLATEAU PROBLEM

LECTURE #17 - Thursday, 31/10 (13:45-15:15)

FUNCTIONS OF BOUNDED VARIATION

Let $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, be open. To motivate the definition of functions of bounded variation, we proved that if $f \in C^2(\Omega)$, then

$$\int_\Omega |\nabla f(x)| \mathrm{d}x = \sup\Big\{ \int_\Omega f(x) \operatorname{div} g(x)\ \mathrm{d}x \ \Big|\ g \in C^1_c(\Omega;\mathbb{R}^n),\ |g| \leq 1 \text{ in } \Omega \Big\}.$$

Since the right hand side makes sense if f is merely in $L^1_{\mathrm{loc}}(\Omega)$, defining the quantity

$$\int_V |Df| := \sup\Big\{ \int_V f \operatorname{div} g \ \mathrm{d}\mathscr{L}^n \ \Big|\ g \in C^1_c(V;\mathbb{R}^n),\ |g| \leq 1 \text{ in } V \Big\}, \qquad V \subseteq \Omega,$$

we introduced:

$\triangleq$ A function $f \in L^1_{\mathrm{loc}}(\Omega)$ is said to be *locally of bounded variation in* Ω, written $f \in BV_{\mathrm{loc}}(\Omega)$, provided $\int_V |Df| < \infty$ for each $V \Subset \Omega$.

$\triangleq$ A function $f \in L^1(\Omega)$ is said to be of *bounded variation in* Ω, written $f \in BV(\Omega)$, provided $\int_\Omega |Df| < \infty$.

After noting some easy properties of BV functions, we proved the following important theorems:

Theorem 3.35 (`semicontinuity`). *Suppose $\{f_k \in BV(\Omega)\}_{k\in\mathbb{N}}$ converges to f in $L^1_{\mathrm{loc}}(\Omega)$. Then*

$$\int_\Omega |Df| \leq \liminf_{k\to\infty} \int_\Omega |Df_k|.$$

Theorem 3.36 (`completeness`). *The set of all $BV(\Omega)$ functions normed by $\|\cdot\|_{BV(\Omega)} := \|\cdot\|_{L^1(\Omega)} + \int_\Omega |D\cdot|$, is a Banach space .*

After stating the following

Theorem 3.37 (`global approximation by smooth functions`). *Assume $f \in BV(\Omega)$. Then there exists a sequence $\{f_k \in C^\infty(\Omega) \cap BV(\Omega)\}_{k\in\mathbb{N}}$ such that $f_k \to f$ in $L^1(\Omega)$ and also $\int_\Omega |\nabla f_k(x)| \mathrm{d}x \to \int_\Omega |Df|$.*

we proved compactness

Theorem 3.38 (`compactness property of` BV). *Let $\Omega \subset \mathbb{R}^n$ be open, bounded with boundary of class $\mathscr{C}^1$. If $\{f_k \in BV(\Omega)\}_{k\in\mathbb{N}}$ is such that $\{\|f_k\|_{BV(\Omega)}\}_{k\in\mathbb{N}}$ is bounded, then there exists $f \in BV(\Omega)$ and a subsequence $\{f_{k_l}\}_{l\in\mathbb{N}}$ such that $f_{k_l} \to f$ in $L^1(\Omega)$.*

SETS OF FINITE PERIMETER

To motivate the definition of sets of finite perimeter, we explained that if $E \subset \mathbb{R}^n$ is a bounded open set with boundary of class $\mathscr{C}^2$, then

$$\int_\Omega |D\chi_E| = \mathscr{H}^{n-1}(\partial E \cap \Omega).$$

We introduced:

$\triangleq$ A measurable set $E \subset \mathbb{R}^n$ is said to be *locally of finite perimeter in* Ω, provided $\chi_E \in BV_{\mathrm{loc}}(\Omega)$.

$\triangleq$ A measurable set $E \subset \mathbb{R}^n$ is said to be of *finite perimeter in* Ω, provided $\chi_E \in BV(\Omega)$.

EXISTENCE OF AREA MINIMIZING SETS

We applied the above results and the direct method to establish the existence of area minimizing sets/minimal surfaces:

Theorem 3.39. *Let Ω be a bounded open set in $\mathbb{R}^n$ and let E be a set of finite perimeter in $\mathbb{R}^n$. Then there exists a set E' that coincides with E outside Ω, such that*

$$\int_{\mathbb{R}^n} |D\chi_{E'}| \leq \int_{\mathbb{R}^n} |D\chi_F| \qquad \text{for all sets } F \text{ that coincide with } E \text{ outside } \Omega.$$

TUTORIAL #07 Thursday, 31/10 (15:30-17:00) We used this time slot also for the Plateau problem presented above.

VIII. MINIMIZERS AND WEAK SOLUTIONS OF THE EULER-LAGRANGE

LECTURE #18 - Monday, 04/11 (15:30-17:00) Recall from V. that if $I[\cdot]$ is given by (6) with L being smooth and $u \in \mathscr{A}_g \cap C^2(U)$ is such that $I[u] = \inf_{w \in \mathscr{A}_g} I[w]$, then u solves the Euler-Lagrange equation (8). If we know instead u is merely in $\mathscr{A}_g$, then we have

Theorem 3.40. *Suppose there exists positive constants C_1, C_2 such that*

$$(H1) \quad |L(\xi,z,x)| \leq C_1(|\xi|^q + |z|^q + 1) \qquad \forall\ (\xi,z,x) \in \mathbb{R}^n \times \mathbb{R} \times U,$$

$$(H2) \quad |\nabla_\xi L(\xi,z,x)|, |L_z(\xi,z,x)| \leq C_2(|\xi|^{q-1} + |z|^{q-1} + 1) \qquad \forall\ (\xi,z,x) \in \mathbb{R}^n \times \mathbb{R} \times U.$$

If $u \in \mathscr{A}_g$ is a minimizer of $I[\cdot]$ on $\mathscr{A}_g$, then u is a `weak solution` *of the boundary value problem associated to* (8)*; that is*

$$\begin{cases} \int_U \left\{\nabla_\xi L(\nabla u(x), u(x), x) \cdot \nabla v(x) + L_z(\nabla u(x), u(x), x) v(x)\right\} \mathrm{d}\mathscr{L}^n(x) = 0 & \forall\ v \in W_0^{1,q}(U), \\ u = g & \text{on } \partial U \text{ in the sense of traces}. \end{cases} \tag{10}$$

Moreover, if L is convex in the (ξ,z)-variables, then any weak solution of the boundary value problem (10)*, is also a minimizer of $I[\cdot]$ on $\mathscr{A}_g$.*

IX. HÖLDER CONTINUITY OF MINIMIZERS

How smooth is a weak solution of (8)? For instance, regardless if a given $u \in W^{1,q}(U)$ is such a solution or not, if $q > n$ then Theorem 3.28 implies $u \in C^{0,1-n/q}_{\mathrm{loc}}(U)$. On the other hand, in case $1 < q \leq n$ a function $u \in W^{1,q}(U)$ may not even be locally bounded. However, if we know additionally that it verifies the first equality in (10) with L satisfying $(H1)$ and $(H2)$, then such a u is $C^{0,\alpha}_{\mathrm{loc}}(U)$ for some $\alpha \in (0,1]$. In the next couple of weeks we prove this fact for the special case where

$$L = L(\xi) = \frac{1}{q}|\xi|^q, \qquad \xi \in \mathbb{R}^n.$$

Note that in this case we have L is convex and $|L(\xi)| = \frac{1}{q}|\xi|^q$ for all $\xi \in \mathbb{R}^n$ (this is $(H1)$ with $C_1 = 1/q$), as well as $\nabla_\xi L(\xi) = |\xi|^{q-2}\xi \Rightarrow |\nabla_\xi L(\xi)| = |\xi|^{q-1}$ for all $\xi \in \mathbb{R}^n$ (this is $(H2)$ with $C_2 = 1$).

LECTURE #19 - Thursday, 07/11 (13:45-15:15) From now on:

$p \in (1,n]$ and Ω is a domain of $\mathbb{R}^n$, $n \in \mathbb{N} \setminus \{1\}$,

and our goal is to prove the following

Theorem 3.41. *Suppose $\tilde{u} \in W^{1,p}(\Omega)$ is a weak solution of the p-Laplace equation in Ω; this means it verifies*

$$\int_\Omega |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla v \, \mathrm{d}\mathscr{L}^n = 0 \qquad \forall\, v \in W_0^{1,p}(\Omega).$$

Then $\tilde{u} \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha = \alpha(n,p) > 0$. In particular, whenever $B_{4r} \Subset \Omega$ we have

$$\sup_{\substack{x,y \in B_r \\ x \neq y}} \frac{|\tilde{u}(x) - \tilde{u}(y)|}{|x-y|^\alpha} \le L = L\big(n, p, \operatorname{osc}_{B_{4r}}(u)\big)$$

For the proof we will need several lemmas. We proved first

Lemma 3.42. *Let U be a bounded domain of $\mathbb{R}^n$ and $f \in L^1(U)$. Assume that for some $q > n$ there exists a positive constant κ such that*

$$\sup_{\rho > 0,\ \omega \in U} \frac{1}{\rho^{n(1-1/q)}} \int_{U \cap B_\rho(\omega)} |f| \, \mathrm{d}\mathscr{L}^n \le \kappa.$$

Then

$$\sup_{x \in U} \left| \int_U \frac{f(z)}{|x-z|^{n-1}} \, \mathrm{d}\mathscr{L}^n(z) \right| \le \kappa n \frac{q-1}{q-n} \big(\operatorname{diam}(U)\big)^{1-n/q}.$$

Combining the above Lemma with Lemma 3.25, we discovered a condition that guarantees local Hölder continuity. More precisely we proved

Lemma 3.43. *Let $u \in W^{1,1}(\Omega)$ and assume that for some $\alpha \in (0,1]$ there exists a positive constant κ such that*

$$\sup_{\rho > 0,\ \omega \in \Omega} \frac{1}{\rho^{n-1+\alpha}} \int_{\Omega \cap B_\rho(\omega)} |\nabla u| \, \mathrm{d}\mathscr{L}^n \le \kappa.$$

Then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$. In particular, we have

$$\sup_{B_{2r}(x_0) \subset \Omega} \ \sup_{x,y \in B_r(x_0)} |u(x) - u(y)| \le c(n,\alpha) \kappa r^\alpha.$$

For the proof we applied Lemma 3.42 with $U = B_r(x_0)$, $q = n/(1-\alpha)$ and $f = |\nabla u|$. Then we applied Hölder's inequality to derive the following generalization of the above lemma.

Lemma 3.44. *Let $u \in W^{1,p}(\Omega)$ and assume that for some $\alpha \in (0,1]$ there exists a positive constant κ such that*

$$\sup_{\rho > 0,\ \omega \in \Omega} \frac{1}{\rho^{n-p+p\alpha}} \int_{\Omega \cap B_\rho(\omega)} |\nabla u|^p \, \mathrm{d}\mathscr{L}^n \le \kappa.$$

Then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$. In particular, we have

$$\sup_{B_{2r}(x_0) \subset \Omega} \ \sup_{x,y \in B_r(x_0)} |u(x) - u(y)| \le c(n,\alpha,p) \kappa^{1/p} r^\alpha.$$

We proved one more consequence of Lemma 3.25:

Lemma 3.45 (`Poincaré inequality on balls`). *Let $u \in W^{1,p}(\Omega)$ and $B_{2r}(x_0) \subset \Omega$. Then*

$$\|u - u_{B_r(x_0)}\|_{L^p(B_r(x_0))} \leq c(n,p) r \, \|\nabla u\|_{L^p(B_r(x_0))}.$$

TUTORIAL #08 Thursday, 07/11 (15:30-17:00) Solutions to exercises - Assignment #08.

LECTURE #20 - Monday, 11/11 (15:30-17:00) We used the *hole-filling* argument of Widman[4] to prove Theorem 3.41 in the case $p = n$. In the proof we used once more a Poincaré inequality, this time for annular domains. We described very briefly (exercise) how this inequality pops out from the following general Poincaré inequality

Theorem 3.46 (`general Poincaré inequality`). *Let $p \geq 1$ and U be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, with boundary of class $\mathscr{C}^1$. Then there exists a positive constant $C_{\mathbf{P}}$, depending only on n, p and U, such that*

$$\|u - u_U\|_{L^p(U)} \leq C_{\mathbf{P}} \|\nabla u\|_{L^p(U)} \qquad \forall\, u \in W^{1,p}(U).$$

We proved the above theorem arguing by contradiction and using Theorem 3.31.

LECTURE #21 - Thursday, 14/11 (13:45-15:15) Fix now $p \in (1,n)$ and let $u \in W^{1,p}(\Omega)$ be a weak `subsolution` of the p-Laplace equation in Ω; that is

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, \mathrm{d}\mathscr{L}^n \leq 0 \qquad \forall\, v \in W_0^{1,p}(\Omega),\ v \geq 0 \text{ a.e. in } \Omega.$$

We proved first the following technical lemma.

Lemma 3.47. *The function $u_+ := \max\{u, 0\} = u\chi_{\{u \geq 0\}}$ is also a weak subsolution of the same equation in Ω.*

The above lemma allowed us to prove the following two lemmata only in the special case where $u \geq 0$. The first one is a reverse weighted Poincaré inequality[5] (the weight being $u^{\beta-1}$):

Lemma 3.48. *Let $\beta > 0$. Then*

$$\int_{\{u>0\}} u^{\beta-1} |\nabla u|^p \eta^p \, \mathrm{d}\mathscr{L}^n \leq C(p)\beta^{-p} \int_{\{u>0\}} u^{p+\beta-1} |\nabla \eta|^p \, \mathrm{d}\mathscr{L}^n \qquad \forall\, \eta \in C_c^\infty(\Omega),\ \eta \geq 0, \tag{11}$$

provided the right hand side is finite[6].

Using the above lemma and Sobolev's inequality we produced a local reverse Hölder inequality. Iterating this last one, we proved

Lemma 3.49. *Let $q > p - 1$. Then*

$$\sup_{B_R} u_+ \leq C(n,p,q) \Big(\fint_{B_{2R}} (u_+)^q \mathrm{d}\mathscr{L}^n \Big)^{1/q} \qquad \forall\, B_{2R} \Subset \Omega, \tag{12}$$

provided the right hand side is finite[7].

Proposition 3.50 (`local boundedness of solutions`). *Under the same assumptions as in Theorem 3.41, $\tilde{u}$ is locally bounded in Ω.*

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[4] K.-O. Widman *Hölder continuity of solutions of elliptic systems*. manuscripta math. **5**, (1971) 299-308

[5] or "Caccioppoli estimate" (weighted)

[6] Since $u \in L^p(\Omega)$, the right hand side is clearly finite at least for all $\beta \in (0,1]$.

[7] Since $u \in L^p(\Omega)$, the right hand side is clearly finite at least for all $q \in (p-1, p]$.

Proof: The solution $\tilde{u}$ is both a weak subsolution and a weak supersolution (see the next lecture for the definition). Hence:

- Since $\tilde{u}$ is a weak subsolution then Lemma 3.47 implies $\tilde{u}_+$ is also a weak subsolution. Applying Lemma 3.49 with $q = p$ we deduce that $\tilde{u}_+$ is locally bounded.
- Since $\tilde{u}$ is a weak supersolution then $-\tilde{u}$ is a weak subsolution and Lemma 3.47 implies $(-\tilde{u})_+$ is also a weak subsolution. Applying Lemma 3.49 with $q = p$ we deduce that $(-\tilde{u})_+$ is locally bounded.

The proposition follows since $|\tilde{u}| = \tilde{u}_+ + \tilde{u}_- = \tilde{u}_+ + (-\tilde{u})_+$. ■

Corollary 3.51. *The right hand sides of* (11) *and* (12) *are finite for any* $\beta > 0$ *and* $q > p-1$ *respectively.*

TUTORIAL #09 Thursday, 14/11 (15:30-17:00) We used this time slot also for the proofs of the above lemmata.

LECTURE #22 - Monday, 18/11 (15:30-17:00) Let now $u \in W^{1,p}(\Omega)$ be a nonnegative weak `supersolution` of the p-Laplace equation in Ω; that is, $u \geq 0$ a.e. in Ω and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, \mathrm{d}\mathscr{L}^n \geq 0 \qquad \forall\, v \in W_0^{1,p}(\Omega),\ v \geq 0 \text{ a.e. in } \Omega.$$

We proved first the following estimate, analogous to that of Lemma 3.48.

Lemma 3.52. *Let* $\beta < 0$. *Then*

$$\int_{\{u>0\}} u^{\beta-1}|\nabla u|^p \eta^p \, \mathrm{d}\mathscr{L}^n \leq C(p)|\beta|^{-p}\int_{\{u>0\}} u^{p+\beta-1}|\nabla\eta|^p \, \mathrm{d}\mathscr{L}^n \qquad \forall\, \eta \in C_c^\infty(\Omega),\ \eta \geq 0.$$

Using the above Lemma we proved

Corollary 3.53. *Let* $q > 0$ *and set* $u_\varepsilon := u + \varepsilon$. *Then*

$$\Big(\fint_{B_{2R}} u_\varepsilon^{-q} \mathrm{d}\mathscr{L}^n\Big)^{-1/q} \leq C(n,p,q)\inf_{B_R} u_\varepsilon \qquad \forall\, B_{2R} \Subset \Omega. \tag{13}$$

Corollary 3.54. *With* u_ε *as defined above, we proved*

$$\int_{B_r} |\nabla(\log u_\varepsilon)| \mathrm{d}\mathscr{L}^n \leq \kappa r^{n-1} \qquad \forall\, B_{2r} \Subset \Omega.$$

We showed how this last corollary implies[8] that

$$\Big(\fint_{B_{2R}} u_\varepsilon^{q} \mathrm{d}\mathscr{L}^n\Big)^{1/q} \leq C(n)\Big(\fint_{B_{2R}} u_\varepsilon^{-q} \mathrm{d}\mathscr{L}^n\Big)^{-1/q},$$

whenever $B_{4R} \Subset \Omega$. Coupling this with (12) and (13) for a nonnegative weak solution u (note here that u_ε is then also a positive solution), we established the `Harnack inequality`:

Theorem 3.55. *Let* $u \in W^{1,p}(\Omega)$ *be a nonnegative solution of the p-Laplace equation in* Ω. *Then there exists a constant* $C > 0$, *not depending on u, such that* $\sup_{B_R} u \leq C \inf_{B_R} u$ *provided that* $B_{4R} \Subset \Omega$.

[8] through the John-Nirenberg lemma

LECTURE #23 - Thursday, 21/11 (13:45-15:35) Using Theorem 3.55 we proved Theorem 3.41. This completed our tour to regularity of minimizers. We solved the exercises of Assignment #09.

X. CONSTRAINED MINIMIZATION PROBLEMS

LECTURE #24 - Monday, 25/11 (15:30-17:00)
We proved the following theorem on existence of a minimizer for the Dirichlet integral together with an integral constraint and on the particular Euler-Lagrange equation it satisfies.

Theorem 3.56. *Let $U \subset \mathbb{R}^n$ be open, bounded with boundary of class $\mathscr{C}^1$ and suppose $G : \mathbb{R} \to \mathbb{R}$ is C^1 with $|G'(z)| \leq C(|z|+1)$ for all $z \in \mathbb{R}$ and some constant $C > 0$. Then the problem*

$$\inf_{w \in \mathscr{A}_G} \left\{ I[w] := \frac{1}{2} \int_U |\nabla w|^2 \, \mathrm{d}\mathscr{L}^n \right\}, \qquad \text{where } \mathscr{A}_G := \left\{ w \in W_0^{1,2}(U) \;\middle|\; \int_U G(w) \, \mathrm{d}\mathscr{L}^n = 0 \right\},$$

admits a solution $u \in \mathscr{A}_G$ provided $\mathscr{A}_G \neq \emptyset$. Assuming further that $G'(u)$ is not equal to zero a.e. within U, then there exists $\lambda \in \mathbb{R}$ such that

$$\int_U \nabla u \cdot \nabla v \, \mathrm{d}\mathscr{L}^n = \lambda \int_U G'(u) v \, \mathrm{d}\mathscr{L}^n \qquad \forall\, v \in W_0^{1,2}(U);$$

that is, u is a weak solution of the boundary value problem

$$\begin{cases} -\Delta u = \lambda G'(u) & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

We discussed (the proof goes by standard arguments we know) the following theorem on constraints of *obstacle* type

Theorem 3.57. *Let $U \subset \mathbb{R}^n$ be open, bounded with boundary of class $\mathscr{C}^1$ and suppose $h : \bar{U} \to \mathbb{R}$ is a given smooth function. Then the problem*

$$\inf_{w \in \mathscr{A}_h} \left\{ I[w] := \int_U \left(\frac{1}{2} |\nabla w|^2 - f w \right) \mathrm{d}\mathscr{L}^n \right\}, \qquad \text{where } \mathscr{A}_h := \left\{ w \in W_0^{1,2}(U) \;\middle|\; w \geq h \text{ a.e. in } U \right\},$$

admits a unique solution $u \in \mathscr{A}_h$ provided $\mathscr{A}_h \neq \emptyset$. Furthermore, u satisfies the following variational inequality

$$\int_U \nabla u \cdot \nabla (w-u) \, \mathrm{d}\mathscr{L}^n \geq \lambda \int_U f(w-u) \, \mathrm{d}\mathscr{L}^n \qquad \forall\, w \in \mathscr{A}_h.$$

XI. THE ISOPERIMETRIC INEQUALITY[9]

LECTURE #25 - Monday, 28/11 (13:45-15:30) The isoperimetric inequality in $\mathbb{R}^n$ asserts that among all smooth bounded *domains* (open and connected sets) of $\mathbb{R}^n$, $n \geq 2$, having the same fixed perimeter (or surface area), it is the ball that maximizes the volume. We have explained in the past why this is expressed through the following theorem.

Theorem 3.58. *Let Ω be a bounded smooth domain in $\mathbb{R}^n$, $n \geq 2$. Then*

$$\mathscr{H}^{n-1}(\partial \Omega) \geq n \omega_n^{1/n} \left[\mathscr{L}^n(\Omega) \right]^{1-1/n},$$

with equality if and only if Ω is a ball.

We gave a proof of the above theorem due to X. Cabré. We also mentioned a version of the isoperimetric inequality that doesn't involve any notion of perimeter.

[9] the interested students can consult the book *Isoperimetric inequalities: Differential geometric and analytic perspectives*, by Isaac Chavel, Cambridge Univ. Press, Cambridge 2001, and also the first chapter of the notes by Manuel Ritoré, that one finds in *Mean Curvature Flow and Isoperimetric Inequalities*-Advanced courses in Mathematics CRM Barcelona, by Manuel Ritoré and Carlo Sinestrari, Birkhauser 2010.