Lehrstuhl für Mathematik IV
Georgios Psaradakis, PhD

Universität Mannheim
Fakultät für Wirtschaftsinformatik
und Wirtschaftsmathematik

Thursday 05/09/2019

Calculus of Variations and Applications - Assignment #01

Exercise 1: Use elementary computations to prove that "among all rectangles with fixed perimeter, it is the square that maximizes the volume".

Exercise 2: Use elementary calculus to show that $|a+b|^p \leq 2^{p-1}\big(|a|^p + |b|^p\big)$ for all $a, b \in \mathbb{R}$, where $p \geq 1$ is fixed.

Exercise 3: Use elementary calculus to prove that the maximum value of the functional

$$\mathscr{F}[u] := \int_0^1 \Big(u'(x)\sin\big(\pi u(x)\big) - \big(x + u(x)\big)^2\Big)\mathrm{d}x, \qquad u \in C^1\big((0,1)\big),$$

is $2/\pi$ for $u(x) = -x$.

Exercise 4: Suppose the continuous function $u : [a,b] \to \mathbb{R}$ is such that

$$\int_a^b u(x)\phi(x)\mathrm{d}x = 0,$$

for all C^1 functions $\phi : [a,b] \to \mathbb{R}$ with $\phi(a) = \phi(b) = 0$. Prove that $u \equiv 0$.

Solutions to be delivered by Thursday 12/09/2019 in class or before 15:30 in Box 46213.

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Thursday 12/09/2019

Calculus of Variations and Applications - Assignment #02

Exercise 1: Suppose $f_k \to f$ in $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$. Prove:

(i) $\lim_{k\to\infty} \|f_k\|_p = \|f\|_p$.

(ii) $f_k \rightharpoonup f$ in $L^p(\mathbb{R}^n)$ (for $p < \infty$).

Exercise 2: Suppose $\lim_{k\to\infty} \|f_k\|_2 = \|f\|_2$. Prove:

(i) If $f_k \rightharpoonup f$ in $L^2(\mathbb{R}^n)$, then $f_k \to f$ in $L^2(\mathbb{R}^n)$.

(ii) If $f_k \to f$ a.e. in $\mathbb{R}^n$, then $f_k \to f$ in $L^2(\mathbb{R}^n)$.

Exercise 3: Let $1 \le p < \infty$ and suppose that $f_k \to f$ in $L^p(\mathbb{R}^n)$. Assume $\{g_k \in L^\infty(\mathbb{R}^n)\}_{k\in\mathbb{N}}$ is such that

(i) $g_k \to g$ a.e. in $\mathbb{R}^n$,

(ii) $\sup_{k\in\mathbb{N}} \|g_k\|_\infty \le M < \infty$.

Prove that $f_k g_k \to fg$ in $L^p(\mathbb{R}^n)$.

Exercise 4: Prove the *interpolation inequality* for L^p-norms: If $1 \le p \le q \le r \le \infty$, $p \ne r$, and $f \in L^p(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ then

$$\|f\|_q \le \|f\|_p^\theta \|f\|_r^{1-\theta}, \quad \text{where } \theta = \frac{1/q - 1/r}{1/p - 1/r}.$$

Exercise 5: The *convolution* of two measurable functions $f, g : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y) g(y) \, \mathrm{d}\mathscr{L}^n(y),$$

for any $x \in \mathbb{R}^n$ such that the integral exists. Use Fubini's theorem and Hölder's inequality to prove the following version of *Young's inequality*:

If $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, $p \in [1, \infty]$, then $f * g \in L^p(\mathbb{R}^n)$ with

$$\|f * g\|_p \le \|f\|_1 \|g\|_p.$$

Solutions to be delivered by Thursday 19/09/2019 in class or before 15:30 in Box 46213.

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Thursday 19/09/2019

Calculus of Variations and Applications - Assignment #03

Exercise 1: Use the idea of the proof of the Poincaré inequality (study first the relevant file uploaded on the webpage of the course) to prove the *Hardy inequality*; that is,

$$\int_0^1 \big(u'(x)\big)^2 \mathrm{d}x \geq \frac{1}{4}\int_0^1 \frac{u^2(x)}{x^2}\mathrm{d}x, \qquad \forall\, u \in C_c^1\big((0,1)\big).$$

[Hint: Find the Euler-Lagrange equation for the corresponding $f = f(x,u,\xi)$ and observe that $x^{1/2}$ is a strictly positive solution of it. Then proceed exactly as in the proof of the Poincaré inequality.]

Exercise 2: We have shown in the first tutorial (as an application of the Fundamental Lemma of the Calculus of Variations) that if $f \in L^1_{\mathrm{loc}}\big((a,b)\big)$ and

$$\int_a^b f\eta\,\mathrm{d}x = 0 \quad \forall\, \eta \in C_c^\infty\big((a,b)\big) \text{ with } \int_a^b \eta\,\mathrm{d}x = 0,$$

then f coincides a.e. in (a,b) with a constant function. Use this to prove that if $f \in L^1_{\mathrm{loc}}\big((a,b)\big)$ and

$$\int_a^b f\eta'\,\mathrm{d}x = 0 \quad \forall\, \eta \in C_c^\infty\big((a,b)\big),$$

then f coincides a.e. in (a,b) with a constant function. What is the answer if you are given instead that

$$\int_a^b f\eta''\,\mathrm{d}x = 0 \quad \forall\, \eta \in C_c^\infty\big((a,b)\big) \ ?$$

Exercise 3: Use Fubini's theorem and Hölder's inequality to prove the following version of Minkowski's inequality

$$\bigg(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^m}|f(x,y)|\mathrm{d}y\Big)^p\mathrm{d}x\bigg)^{1/p} \leq \int_{\mathbb{R}^m}\Big(\int_{\mathbb{R}^n}|f(x,y)|^p\mathrm{d}x\Big)^{1/p}\mathrm{d}y,$$

for all $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ which are (let's say) continuous with compact support.

Solutions to be delivered by Thursday 26/09/2019 in class or before 15:30 in Box 46213.

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Friday 27/09/2019

Calculus of Variations and Applications - Assignment #04

Exercise 1: **A)** Let $V \in C(\Omega)$ where $\Omega \subseteq \mathbb{R}^n$ is open, $n \in \mathbb{N}$. Show that if $f \in C^2(\Omega)$ is a strictly positive (or strictly negative) solution of the equation

$$\Delta f(x) + V(x) f(x) = 0 \qquad x \in \Omega, \tag{1}$$

then the following generalized Hardy inequality is true:

$$\int_\Omega |\nabla u(x)|^2 \mathrm{d}x \geq \int_\Omega V(x) u^2(x) \mathrm{d}x \qquad \text{for all } u \in C_c^\infty(\Omega).$$

[Hint: Apply the ground state transform (exactly as in the proof of Poincaré's inequality or Exercise 1 of the previous problem set).]

B) Let $\mathbb{R}^n_+ := \{x = (x_1, ..., x_n) \in \mathbb{R}^n \mid x_n > 0\}$. Prove

$$\int_{\mathbb{R}^n_+} |\nabla u(x)|^2 \mathrm{d}x \geq \frac{1}{4} \int_{\mathbb{R}^n_+} \frac{u^2(x)}{x_n^2} \mathrm{d}x, \qquad \text{for all } u \in C_c^\infty(\mathbb{R}^n_+).$$

[Hint: From A), it suffices to find a positive solution f of (1) with $V = \frac{1}{4} x_n^{-2}$ and $\Omega = \mathbb{R}^n_+$. To this end, try $f(x) = x_n^\alpha$ and find α.]

C) Let $\mathbb{R}^n_\star := \mathbb{R}^n \setminus \{0\}$, $n \in \mathbb{N} \setminus \{1, 2\}$. Prove

$$\int_{\mathbb{R}^n_\star} |\nabla u(x)|^2 \mathrm{d}x \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n_\star} \frac{u^2(x)}{|x|^2} \mathrm{d}x \qquad \text{for all } u \in C_c^\infty(\mathbb{R}^n_\star).$$

[Hint: From A), it suffices to find a positive solution f of (1) with $V = \left(\frac{n-2}{2}\right)^2 |x|^{-2}$ and $\Omega = \mathbb{R}^n_\star$. To this end, try $f(x) = |x|^\alpha$ and find α.]

Solutions to be delivered by Thursday 07/10/2019 in class or before 15:30 in Box 46213.

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Thursday 10/10/2019

Calculus of Variations and Applications - Assignment #05

Exercise 1: Recall that we write $V \Subset U$ whenever U, V be open subsets of $\mathbb{R}^n$ such that $\bar{V} \subset U$ and $\bar{V}$ is compact. Show that if $V \Subset U$ then there exists $\zeta \in C_c^\infty(\mathbb{R}^n)$ such that support$\{\zeta\} \subset U$, $0 \le \zeta \le 1$ in U and $\zeta \equiv 1$ on V.

Exercise 2: Let $p \ge 1$ and U be an open subset of $\mathbb{R}^n$, $n \in \mathbb{N} \setminus \{1\}$, such that $\mathscr{L}^n(U) < \infty$. Prove the following version of the Poincaré inequality:

$$\|u\|_{L^p(U)} \le C(n,p)\big(\mathscr{L}^n(U)\big)^{1/n}\|\nabla u\|_{L^p(U)} \qquad \text{for all } u \in C_c^\infty(U).$$

[Hint: For all sufficiently large p, you can use first the Sobolev inequality that we mentioned in the tutorial; that is

$$\|u\|_{L^{q^\star}(U)} \le C(n,q)\|\nabla u\|_{L^q(U)}, \quad q^\star := nq/(n-q), \quad 1 \le q < n,$$

with suitable exponent q such that $q^\star = p$, and then Holder's inequality. For the rest p's simply use first Hölder's and then Sobolev's inequality.]

In the next exercise we write N_u for the zero set of a given function $u : U \to \bar{\mathbb{R}}$; that is,

$$N_u := \{x \in U \mid u(x) = 0\}.$$

Exercise 3: Let $p \ge 1$ and U be a bounded, open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, having smooth boundary. Consider a function $u \in W^{1,p}(U)$ such that

$$\Big\|\frac{1}{u}\Big\|_{L^\alpha(U)} < \infty \qquad \text{for some } \alpha \in (0,\infty).$$

The above assumption readily implies that N_u cannot have positive Lebesgue measure. Show in particular that $N_u = \emptyset$, provided that

$$\frac{1}{n} - \frac{1}{p} \ge \frac{1}{\alpha}.$$

[Hint: Assume there exists $x_0 \in U$ with $u(x_0) = 0$ and use Morrey's inequality to reach a contradiction.]

Solutions to be delivered by Thursday 17/10/2019 in class or before 13:45 in Box 46213.

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Thursday 17/10/2019

Calculus of Variations and Applications - Assignment #06

Exercise 2: Let $u \in L^p(U)$ for some $p \in [1,\infty]$ and $U \subseteq \mathbb{R}^n$ open. Denoting by η_ε the standard mollifier, set as usual $u_\varepsilon(x) := \big(\eta_\varepsilon \star u\big)(x)$, $x \in U_\varepsilon := \{x \in U \mid \operatorname{dist}(x, \partial U) > \varepsilon\}$ if $U \subsetneq \mathbb{R}^n$, or $U_\varepsilon = \mathbb{R}^n$ if $U = \mathbb{R}^n$. Prove that $\|u_\varepsilon\|_{L^p(U_\varepsilon)} \leq \|u\|_{L^p(U)}$.

Exercise 2: Let $U \subset \mathbb{R}^n$, $n \geq 2$, be open with $\mathscr{L}^n(U) < \infty$. Prove that for $1 \leq p < n$ we have $W_0^{1,p}(U) \subset L^q(U)$ for all $q \in [1, np/(n-p)]$, with the estimate

$$\|u\|_{L^q(U)} \leq C(n,p,U)\|\nabla u\|_{L^p(U)} \qquad \text{for all } u \in W_0^{1,p}(U).$$

[Hint: Because of Hölder's inequality it suffices to show this only for $q = np/(n-p)$.]

Exercise 3: Let $U \subset \mathbb{R}^n$, $n \geq 2$, be open, bounded with boundary of class $\mathscr{C}^1$. Prove that for $1 \leq p < n$ we have $W^{1,p}(U) \subset L^q(U)$ for all $q \in [1, np/(n-p)]$, with the estimate

$$\|u\|_{L^q(U)} \leq C(n,p,U)\|u\|_{W^{1,p}(U)} \qquad \text{for all } u \in W^{1,p}(U).$$

[Hint: Because of Hölder's inequality it suffices to show this only for $q = np/(n-p)$. To this end proceed exactly as in the proof of Theorem 3.28, but using the Sobolev inequality in place of Morrey's inequality.]

Solutions to be delivered by Thursday 24/10/2019 in class or before 13:45 in Box 46213.

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Thursday 24/10/2019

Calculus of Variations and Applications - Assignment #07

Exercise 1: Let $\Omega \subset \mathbb{R}^n$ be open. Define for $u \in L^1(\Omega)$ the *variation of* u *in* Ω by

$$V(u;\Omega) := \sup\Big\{\Big|\int_\Omega u \,\mathrm{div}\varphi \,\mathrm{d}\mathscr{L}^n\Big| \;:\; \varphi \in C_c^1(\Omega;\mathbb{R}^n) \text{ with } \|\varphi\|_{L^\infty(\Omega)} \leq 1\Big\}.$$

We define the space of functions of bounded variation as

$$BV(\Omega) := \big\{u \in L^1(\Omega) \;:\; V(u;\Omega) < \infty\big\}.$$

Prove that

(i) $W^{1,1}(\Omega) \subset BV(\Omega)$.

(ii) Prove the above inclusion is strict. To do so take $\Omega = (-1,1)$ (hence $n = 1$) and consider the function defined by $H(x) = 1$ if $x \in (0,1)$, $H(x) = 0$ if $x \in (-1,0)$. Show then that $H \in BV(\Omega)$ but $H \notin W^{1,1}(\Omega)$.

Solutions to be delivered by Thursday 31/10/2019 in class or before 13:45 in Box 46213.

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Friday 01/11/2019

Calculus of Variations and Applications - Assignment #08

Exercise 1: Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex; that is

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda) f(y) \qquad \forall\, \lambda \in [0,1],\ x, y \in \mathbb{R}^n.$$

Prove the following facts:

(i) For $\varepsilon > 0$ denote by η_ε the standard mollifier. Then the mollified function $f_\varepsilon := \eta_\varepsilon \star f$ is also convex.

(ii) If $f \in C^1(\mathbb{R}^n)$, then we have $f(y) \geq f(x) + \nabla f(x) \cdot (y - x)$ for all $x, y \in \mathbb{R}^n$.

(iii) If $f \in C^2(\mathbb{R}^n)$, then the Hessian matrix $D^2 f$ is a nonnegative definite symmetric matrix on $\mathbb{R}^n$; that is $\xi^T \cdot D^2 f(x) \cdot \xi \geq 0$ for any $\xi \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, or what is the same,

$$\sum_{i,j=1}^{n} f_{x_i x_j}(x) \xi_i \xi_j \geq 0 \qquad \forall\, \xi \in \mathbb{R}^n,\ x \in \mathbb{R}^n.$$

Exercise 2: Accept the following fact on global approximation of $BV(\mathbb{R}^n)$ functions by smooth functions with compact support:

> if $f \in BV(\mathbb{R}^n)$, then there exists a sequence $\{f_k \in C_c^\infty(\mathbb{R}^n) \cap BV(\mathbb{R}^n)\}_{k\mathbb{N}}$ such that $f_k \to f$ in $L^1(\mathbb{R}^n)$ and also $\int_{\mathbb{R}^n} |\nabla f_k(x)| \mathrm{d}x \to \int_{\mathbb{R}^n} |Df|$.

Use this to prove the Gagliardo-Nirenberg inequality in BV; that is

$$\|f\|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq C(n) \int_{\mathbb{R}^n} |Df| \qquad \forall\, f \in BV(\mathbb{R}^n),$$

where $C(n)$ is the same constant as in Gagliardo-Nirenberg inequality (see displayed formula (5) on page 6 of the course calendar).

Solutions to be delivered by Thursday 07/11/2019 in class or before 13:45 in Box 46213.

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Thursday 14/11/2019

Calculus of Variations and Applications - Assignment #09

Exercise 1: Provide the details on how the Poincaré inequality in a bounded smooth domain $U \subset \mathbb{R}^n$:

$$\|u - u_U\|_{L^p(U)} \leq C(n,p,U)\|\nabla u\|_{L^p(U)} \qquad \forall\, u \in W^{1,p}(U),$$

implies the Poincaré inequality in the annular domain $A_r := B_{2r}(x_0) \setminus B_r(x_0)$ (some $x_0 \in \mathbb{R}^n$)

$$\|u - u_{A_r}\|_{L^p(A_r)} \leq C(n,p)r\|\nabla u\|_{L^p(A_r)} \qquad \forall\, u \in W^{1,p}(A_r).$$

Exercise 2: Suppose $U \subset \mathbb{R}^n$ satisfies $0 < \mathscr{L}^n(U) < \infty$. If u is a measurable function with $|u|^p \in L^1(U)$ for some $p \in \mathbb{R}$, we define

$$\Phi_p(u) := \left(\frac{1}{\mathscr{L}^n(U)}\int_U |u|^p \mathrm{d}\mathscr{L}^n\right)^{1/p}.$$

Prove:

I. $\Phi_p(u) \leq \Phi_q(u)$ whenever $1 \leq p < q < \infty$,

II. $\lim_{p\to\infty} \Phi_p(u) = \sup_U |u|$, and

III. $\lim_{p\to-\infty} \Phi_p(u) = \inf_U |u|$.

Solutions to be delivered by Thursday 21/11/2019 in class or before 13:45 in Box 46213.