Some preliminary material for the course "Calculus of Variations & Applications"

0.1 Lebesgue measure

► For $n \in \mathbb{N}$, an n-dimensional *interval* I is a subset of $\mathbb{R}^n$ of the form

$$I = \big\{x = (x_1, ..., x_n) \ : \ \alpha_k \leq x_k \leq \beta_k, \ k = 1, ..., n\big\}$$
$$\equiv [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times ... \times [\alpha_n, \beta_n],$$

where $\alpha_k < \beta_k$ for all $k = 1, ..., n$. The *volume of an interval* I is given by

$$\mathrm{Vol}(I) = \prod_{i=1}^{n} (\beta_k - \alpha_k).$$

► The *Lebesgue measure* (denoted throughout by $\mathscr{L}^n$), is defined for any $A \subseteq \mathbb{R}^n$ by

$$\mathscr{L}^n(A) := \inf\Big\{\sum_{k\in\mathbb{N}} \mathrm{Vol}(I_k) : \{I_k\}_{k\in\mathbb{N}} \text{ are intervals and } A \subseteq \bigcup_{k\in\mathbb{N}} I_k\Big\}.$$

One can show for an interval I that $\mathscr{L}^n(I) = \mathscr{L}^n(I^\circ) = \mathrm{Vol}(I)$.

► A set $A \subseteq \mathbb{R}^n$ is called $\mathscr{L}^n$*-measurable* (from now on only *measurable*) if

$$\mathscr{L}^n(B) = \mathscr{L}^n(B \cap A) + \mathscr{L}^n(B \setminus A) \quad \forall\, B \subseteq \mathbb{R}^n.$$

Hence, a set $A \subseteq \mathbb{R}^n$ is measurable if and only if the set $A^c := \mathbb{R}^n \setminus A$ is measurable. Also, $\emptyset$, $\mathbb{R}^n$ and sets of measure 0 are measurable. It turns out that if $\{A_k \subset \mathbb{R}^n\}_{k\in\mathbb{N}}$ is a sequence of measurable sets, then

(i) if $\{A_k\}_{k\in\mathbb{N}}$ are disjoint, then $\mathscr{L}^n\big(\bigcup_{k\in\mathbb{N}} A_k\big) = \sum_{k\in\mathbb{N}} \mathscr{L}^n(A_k)$,

(ii) if $\{A_k\}_{k\in\mathbb{N}}$ is non-decreasing, then $\lim_{k\to\infty} \mathscr{L}^n(A_k) = \mathscr{L}^n\big(\bigcup_{k\in\mathbb{N}} A_k\big)$,

(iii) if $\{A_k\}_{k\in\mathbb{N}}$ is non-increasing and $\mathscr{L}^n(A_1) < \infty$, then $\lim_{k\to\infty} \mathscr{L}^n(A_k) = \mathscr{L}^n\big(\bigcap_{k\in\mathbb{N}} A_k\big)$,

(iv) the sets $\bigcup_{k\in\mathbb{N}} A_k$ and $\bigcap_{k\in\mathbb{N}} A_k$ are measurable.

► A collection of subsets $\mathscr{A} \subseteq 2^{\mathbb{R}^n}$ is called a *σ-algebra* provided

$$\text{(i) } \emptyset,\ \mathbb{R}^n \in \mathscr{A}, \qquad \text{(ii) } A \in \mathscr{A} \Rightarrow \mathbb{R}^n \setminus A \in \mathscr{A}, \qquad \text{(iii) } \{A_k \in \mathscr{A}\}_{k\in\mathbb{N}} \Rightarrow \bigcup_{k\in\mathbb{N}} A_k \in \mathscr{A}.$$

Hence, the collection of all measurable sets form a σ-algebra. The *Borel σ-algebra* of $\mathbb{R}^n$ is defined as the smallest σ-algebra of $\mathbb{R}^n$ containing the open subsets of $\mathbb{R}^n$. A set in the Borel σ-algebra will be called a *Borel set*.

► The Lebesgue measure is a *Radon measure on* $\mathbb{R}^n$; that means it satisfies the following properties

(i) every Borel set is measurable,

(ii) for each $C \subseteq \mathbb{R}^n$ there exists Borel set B such that $C \subseteq B$ and $\mathscr{L}^n(C) = \mathscr{L}^n(B)$,

(iii) $\mathscr{L}^n(K) < \infty$ for all compact $K \subset \mathbb{R}^n$.

► The Lebesgue measure has the following property (the *Brunn-Minkowski inequality*):

$$\big(\mathscr{L}^n(A+B)\big)^{1/n} \geq \big(\mathscr{L}^n(A)\big)^{1/n} + \big(\mathscr{L}^n(B)\big)^{1/n} \qquad \forall\, A,\ B \subseteq \mathbb{R}^n,$$

where $A+B$ stands for the Minkowski sum of A, B.

0.2 Lebesgue measurable functions

► A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called *$\mathscr{L}^n$-measurable* (from now on only *measurable*) if

$$U \subseteq \mathbb{R}^m \text{ is open } \Rightarrow f^{-1}(U) \text{ is measurable.}$$

If $f, g : \mathbb{R}^n \to \bar{\mathbb{R}}$ are measurable, then so are $f+g$, fg, $|f|$, $\min\{f,g\}$, $\max\{f,g\}$ and f/g (provided $g \neq 0$ in $\mathbb{R}^n$). Also, if $\{f_k : \mathbb{R}^n \to \bar{\mathbb{R}}\}_{k\in\mathbb{N}}$ are measurable, then so are $\inf_{k\in\mathbb{N}} f_k$, $\sup_{k\in\mathbb{N}} f_k$, $\liminf_{k\to\infty} f_k$ and $\limsup_{k\to\infty} f_k$.

► A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called *Borel measurable* if

$$U \subseteq \mathbb{R}^m \text{ is open } \Rightarrow f^{-1}(U) \text{ is a Borel set.}$$

Hence, if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous then it is Borel measurable (recall that the pre-image $f^{-1}(U)$ of an open set $U \subseteq \mathbb{R}^m$ through a continuous function f is again open).

► If $f : \mathbb{R}^n \to [0,\infty]$ is measurable, there exist measurable sets $\{A_k \subseteq \mathbb{R}^n\}_{k\in\mathbb{N}}$ such that

$$f(x) = \sum_{k\in\mathbb{N}} \frac{1}{k} \chi_{A_k}(x) \quad \forall\, x \in \mathbb{R}^n.$$

► [**Lusin's theorem**] Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be measurable and $A \subset \mathbb{R}^n$ be measurable with $\mathscr{L}^n(A) < \infty$. Then

$$\forall\, \varepsilon > 0,\ \ \exists \text{ compact } K \subset A \text{ such that } \mathscr{L}^n(A \setminus K) < \varepsilon \text{ and } f|_K \text{ is continuous.}$$

► The expression "a.e. in A" where $A \subseteq \mathbb{R}^n$ means "almost everywhere in A with respect to $\mathscr{L}^n$"; that is, "for all $x \in A \setminus N$ where $\mathscr{L}^n(N) = 0$".

► [**Egoroff's theorem**] Let $\{f_k : \mathbb{R}^n \to \mathbb{R}^m\}_{k\in\mathbb{N}}$ be measurable such that $f_k \to f$ a.e. in A, where $A \subset \mathbb{R}^n$ is measurable with $\mathscr{L}^n(A) < \infty$. Then

$$\forall\, \varepsilon > 0,\ \ \exists \text{ measurable } B \subset A \text{ such that } \mathscr{L}^n(A \setminus B) < \varepsilon \text{ and } f_k \to f \text{ uniformly in } B.$$

0.3 Lebesgue integral

► For a function $g : \mathbb{R}^n \to \bar{\mathbb{R}}$ we set $g_+ := \max\{0, g\}$ and $g_- := \max\{0, -g\}$. Observe these are nonnegative functions and that the following decompositions are valid: $g = g_+ - g_-$ and $|g| = g_+ + g_-$.

► A function $g : \mathbb{R}^n \to \bar{\mathbb{R}}$ is called *simple function* if the image of g is countable. Hence, if $g : \mathbb{R}^n \to \bar{\mathbb{R}}$ is simple, then there exist disjoint $\{A_k \subseteq \mathbb{R}^n\}_{k\in\mathbb{N}}$ and $\{\alpha_k \in \bar{\mathbb{R}}\}_{k\in\mathbb{N}}$ such that

$$g = \sum_{k\in\mathbb{N}} \alpha_k \chi_{A_k} \quad \text{and} \quad \bigcup_{k\in\mathbb{N}} A_k = \mathbb{R}^n.$$

Furthermore, $\{A_k\}_{k\in\mathbb{N}}$ can be taken measurable if g is known to be measurable.

► Thus, given a measurable simple function $g : \mathbb{R}^n \to [0, \infty]$, then there exist disjoint measurable sets $\{A_k\}_{k\in\mathbb{N}}$ and $\{\alpha_k \in (0, \infty]\}_{k\in\mathbb{N}}$ such that

$$g = \sum_{k\in\mathbb{N}} \alpha_k \chi_{A_k} \quad \text{and} \quad \bigcup_{k\in\mathbb{N}} A_k \subseteq \mathbb{R}^n,$$

and so we define its *integral* on $\mathbb{R}^n$ by

$$\int_{\mathbb{R}^n} g \, \mathrm{d}\mathscr{L}^n := \sum_{k\in\mathbb{N}} \alpha_k \mathscr{L}^n(A_k).$$

► If $g : \mathbb{R}^n \to \bar{\mathbb{R}}$ is a measurable simple function and either $\int_{\mathbb{R}^n} g_+ \, \mathrm{d}\mathscr{L}^n < \infty$ or $\int_{\mathbb{R}^n} g_- \, \mathrm{d}\mathscr{L}^n < \infty$, we call g an *integrable simple function* and define its integral on $\mathbb{R}^n$ by

$$\int_{\mathbb{R}^n} g \, \mathrm{d}\mathscr{L}^n := \int_{\mathbb{R}^n} g_+ \, \mathrm{d}\mathscr{L}^n - \int_{\mathbb{R}^n} g_- \, \mathrm{d}\mathscr{L}^n.$$

► Given $f:\mathbb{R}^n \to \bar{\mathbb{R}}$, we define the *upper integral* of f on $\mathbb{R}^n$ by

$$\overline{\int_{\mathbb{R}^n}} f \, d\mathscr{L}^n := \inf\Big\{ \int_{\mathbb{R}^n} g \, d\mathscr{L}^n : g \text{ integrable simple function with } g \geq f \text{ a.e. in } \mathbb{R}^n \Big\}.$$

Correspondingly, the *lower integral* of f on $\mathbb{R}^n$ is defined by

$$\underline{\int_{\mathbb{R}^n}} f \, d\mathscr{L}^n := \sup\Big\{ \int_{\mathbb{R}^n} g \, d\mathscr{L}^n : g \text{ integrable simple function with } g \leq f \text{ a.e. in } \mathbb{R}^n \Big\}.$$

A measurable function $f:\mathbb{R}^n \to \bar{\mathbb{R}}$ is called $\mathscr{L}^n$*-integrable* (from now on only integrable) if $\overline{\int_{\mathbb{R}^n}} f \, d\mathscr{L}^n = \underline{\int_{\mathbb{R}^n}} f \, d\mathscr{L}^n$. We write then $\int_{\mathbb{R}^n} f \, d\mathscr{L}^n$ for their common value.

► It turns out that any nonnegative measurable function is integrable. Also for any integrable f we have $|\int_{\mathbb{R}^n} f \, d\mathscr{L}^n| \leq \int_{\mathbb{R}^n} |f| \, d\mathscr{L}^n$.

► A function $f:\mathbb{R}^n \to \bar{\mathbb{R}}$ is called $\mathscr{L}^n$*-summable* (from now on only *summable*) if it is integrable and $\int_{\mathbb{R}^n} |f| d\mathscr{L}^n < \infty$.

► [**Fatou's lemma**] Let $f_k : \mathbb{R}^n \to [0,\infty]$, $k \in \mathbb{N}$, be measurable. Then

$$\int_{\mathbb{R}^n} \liminf_{k\to\infty} f_k \, d\mathscr{L}^n \leq \liminf_{k\to\infty} \int_{\mathbb{R}^n} f_k \, d\mathscr{L}^n.$$

► [**Monotone convergence theorem**] Let $f_k : \mathbb{R}^n \to [0,\infty]$, $k \in \mathbb{N}$ be measurable such that $f_k \leq f_{k+1}$ a.e. in $\mathbb{R}^n$, for all $k \in \mathbb{N}$. Then

$$\int_{\mathbb{R}^n} \lim_{k\to\infty} f_k \, d\mathscr{L}^n = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k \, d\mathscr{L}^n.$$

► [**Dominated convergence theorem**] Let $f, f_k : \mathbb{R}^n \to \bar{\mathbb{R}}$, $k \in \mathbb{N}$, be measurable, $g : \mathbb{R}^n \to [0,\infty]$ be summable, satisfying

(i) $f_k \to f$ as $k \to \infty$, a.e. in $\mathbb{R}^n$,

(ii) $|f_k| \leq g$ a.e. in $\mathbb{R}^n$, for all $k \in \mathbb{N}$.

Then

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |f_k - f| d\mathscr{L}^n = 0.$$

► [**Absolute continuity of the integral**] If $f:\mathbb{R}^n \to \bar{\mathbb{R}}$ is summable then

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0 \text{ such that if } A \subset \mathbb{R}^n \text{ is measurable with } \mathscr{L}^n(A) < \delta, \text{ then } \int_A |f| d\mathscr{L}^n < \varepsilon.$$

► [$C_c(\mathbb{R}^n)$ **is dense in** $L^1(\mathbb{R}^n)$] If $f:\mathbb{R}^n \to \bar{\mathbb{R}}$ is summable then

$$\forall\, \varepsilon > 0,\ \exists\, g \in C_c(\mathbb{R}^n) \text{ such that } \int_{\mathbb{R}^n} |f - g| \, d\mathscr{L}^n < \varepsilon.$$

► [**a.e. convergence for a subsequence**] Let $f, f_k : \mathbb{R}^n \to \bar{\mathbb{R}}$, $k \in \mathbb{N}$, be summable and $\int_{\mathbb{R}^n} |f_k - f| d\mathscr{L}^n = 0$. Then there exists a subsequence $\{f_{l_k}\}_{k\in\mathbb{N}}$ which converges to f a.e. in $\mathbb{R}^n$.