LETCURE NOTES FOR INTRODUCTION TO PDES

LI CHEN

This is an introductory course for partial differential equations. Four basic types of PDEs, i.e. the transport equation, wave equation, heat equation, and Poisson equation, will be studied. Derivation of the solution formula by using characteristic method, separation of variable (Fourier series), and Fourier transformation are going to be introduced for specific equations, the technics can be used also for other type of equations. Classical tools such as maximum principle and energy estimates are introduced in order to get uniqueness and stability of the solutions.

Contents

Date: HWS2019.
FOR INTERNAL USE ONLY

1. Introduction

1.1. **Definition and examples of linear and nonlinear equations.** Within this lecture notes, we will use the following notations.

Ω is an open subset in $\mathbb{R}^n$. Let u be a smooth function defined on Ω. A **multi-index** is an n dimensional vector, i.e., $\alpha = (\alpha_1, \cdots, \alpha_n)$ and the differential operator D^α is defined by

$$D^\alpha u = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n} u.$$

The order of the operator D^α is given by $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, i.e., D^α is an $|\alpha|$-th order differential operator.

F is a mapping in the following sense

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}.$$

Symbolically a k-th order partial differential equation can be written in the form of

$$F(D^k u(x), D^{k-1} u(x), \cdots, Du(x), u(x), x) = 0, \quad \forall x \in \Omega, \tag{1.1}$$

where D^k are all the k-th order differential operators, i.e., D^α with $|\alpha| = k$. The unknown in the equation is $u(x) : \Omega \to \mathbb{R}$.

Moreover, if $\mathbf{F}$ is a vector-valued function such that

$$\mathbf{F} : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \cdots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m,$$

then the corresponding equation is a k-th order partial differential system

$$\mathbf{F}(D^k \mathbf{u}(x), D^{k-1} \mathbf{u}(x), \cdots, D\mathbf{u}(x), \mathbf{u}(x), x) = \mathbf{0}, \quad \forall x \in \Omega \tag{1.2}$$

and $\mathbf{u} : \Omega \to \mathbb{R}^m$ is the unknown.

If the equation can be written as

$$\sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u = f(x) \tag{1.3}$$

with $a_\alpha(x)$ and $f(x)$ being given functions, then it is called a **linear** equation.

If the equation can be written as

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u + a_0(D^{k-1} u, \cdots, Du, u, x) = 0 \tag{1.4}$$

with $a_\alpha(x)$ being a given function and a_0 being a given mapping, then it is called a **semi-linear** equation.

If the equation can be written as

$$\sum_{|\alpha| = k} a_\alpha(D^{k-1} u, \cdots, Du, u, x) D^\alpha u + a_0(D^{k-1} u, \cdots, Du, u, x) = 0 \tag{1.5}$$

with a_α and a_0 being given mappings, then it is called a **quasilinear** equation.

A PDE is called **fully nonlinear** if the highest order derivative is nonlinear.

Theorem 1.1. *(Superposition principle for homogeneous linear equations). Let $f \equiv 0$ in* (1.3)*. If u_1 and u_2 are both solutions of* (1.3)*, then all the linear combinations of u_1 and u_2 are still solutions of* (1.3)*.*

We list here some examples of partial differential equations.

Linear PDEs

(1) Linear transport equation $u_t + \mathbf{b} \cdot \nabla u = f$,
(2) Wave equation $u_{tt} - \Delta u = f$,
(3) Heat equation $u_t - \Delta u = f$,
(4) Poisson equation $-\Delta u = f$,
(5) Schrödinger equation $iu_t = -\Delta u$,
(6) $\cdots\cdots$.

where we use the notation $\nabla = (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n})$ for n-dimensional gradient operator and $\Delta = \nabla \cdot \nabla = \partial_{x_1x_1} + \partial_{x_2x_2} + \cdots + \partial_{x_nx_n}$ the n-dimensional Laplacian.

There are also some other higher order linear equations, which are mostly from physics (see more examples in Evan's book).

Nonlinear PDEs There are plenty of nonlinear equations in the literature. But we can barely include them in this course. However, if time allows, we will give a short introduction on the 1-D scalar first order hyperbolic conservation laws.

(1) Scalar conservation law $u_t + \mathrm{div}\mathbf{F}(u) = 0$,
(2) Hamilton-Jacobi equation $u_t + H(Du, x) = 0$,
(3) Nonlinear Poisson equation $-\Delta u = f(u)$,
(4) Monge-Ampère equation $\det D^2 u = f$,
(5) Porous medium equation $u_t - \Delta u^m = 0$,
(6) Nonlinear Schrödinger equation $iu_t = -\Delta u + |u|^2 u$,
(7) $\cdots\cdots$.

PDE systems are as well important in the literature, such as Euler system, Navier-Stokes system, Maxwell's equations, reaction diffusion system, multi-component diffusion system, Keller-Segel system. However, due to the limitation of time, we are not going to investigate any system in this course.

1.2. **Setup of the problems.** In order to set up a well-posed PDE problem, additional boundary and initial conditions need to be given.

For time evolutionary PDE, some initial data of the problem is desired. For example, for heat equation $u_t - \Delta u = f$, we need initial conditions like $u|_{t=0} = u_0(x)$ where $u_0(x)$, is a given function. For wave equation $u_{tt} - \Delta u = f$, we need $u|_{t=0} = g$ and $u_t|_{t=0} = h$ since we have double derivatives in time t.

If we study the PDE with space variable x in Ω, which is an open subset in $\mathbb{R}^n$, and $\partial\Omega \neq \emptyset$, we need to give boundary conditions. There are three kinds of typical boundary conditions, each of which has its own physical background.

(1) **Dirichlet boundary condition**, the unknown itself is given on the boundary

$$u|_{\partial\Omega} = u_D(x),$$

where $u_D(x)$ is a given function defined on $\partial\Omega$.

(2) **Neumann boundary condition**, the normal derivative of the unknown is given on the boundary
$$\nabla u \cdot \gamma|_{\partial\Omega} = u_N(x),$$
where γ is the pointwise unit outer normal vector of $\partial\Omega$.

(3) **Robin boundary condition**, the nontrivial linear combination of the unknown and its normal derivative is given on the boundary
$$\alpha u + \beta \nabla u \cdot \gamma|_{\partial\Omega} = u_R(x),$$
where $\alpha(x)$ and $\beta(x)$ are nonnegative functions given on the boundary.

1.3. **Basic topics in studying PDE problems.** After settling down a reasonable PDE problem, our main purpose is to find the solution and to study it in order to have a more clear understanding of the solution behavior. A direct way is to find an analytical solution representation, which gives explicitly how the solution behaves. However, in most of the cases, it is impossible to find an explicit formula for the solution. What can we do then? In working with the problems directly, we can still get detailed information of the solutions without knowing the explict representation.

Given a PDE problem, we need to study its well-posedness, including

(1) **Existence**. As mentioned before that the direct way to get existence of solution is to solve it explicitly. Then under appropriate regularity assumption of the given data, one can show that the existence of classical solution is given by solution formula. Here classical solution means that the k-th order derivatives of the solution exist if it is a solution to a k-th order PDE. For those equations, for which it is hopeless to get solution formula, there are several other ways to get existence, for example, fixed point theorems from functional analysis, variational methods, approximation by approximated solutions are oftenly used in the literature. The main thing one should keep in mind is, in which function class one could expect the existence of solution. Many problems have no classical solution. Usually the existence should be studied in the sense that the equation is satisfied in some weaker formulation instead of its classical one. Afterwards, one can try to find out whether the weak solution has further regularities if the given data are correspondingly smooth.

(2) **Uniqueness**. Once we have existence at hand, a natural question to ask is whether it is unique and of course in which class it is unique. For some of the problems one may also expect to get existence of multi solutions. The larger the existence class, the less the hope to assure uniqueness.

(3) **Stability**. The stability of a solution means that if the given data are slightly perturbed, is it true that the solution also doesn't change so much? In other words, stability means the continuous dependence of the solution on its given data, including initial and boundary conditions, or any other given parameters in the equation.

(4) **Solution behavior**. From the aspect of applications, solution behavior is the most important thing that the PDE analysis should contribute to. Here the solution

behavior we talk about is in the very general sense. It could be the large time behavior, boundary layer behavior or singular behavior if one considers the singular limit problem.

In the literature, the research in PDEs starts from finding explicit solutions. The basic tools that we will discuss in this course are

(1) method of characteristics,
(2) Fourier transform,
(3) separation of variables (Fourier series),
(4) Green's function.

Although these tools to find solution formula couldn't be directly used in many of the PDE problems, they still play a very important role in understanding the basic theories and phenomena.

There are also many other tools to study the solution behavior without using formula, to name just a few,

(1) energy method,
(2) maximum principle,
(3) asymptotic expansion,
(4) $\cdots\cdots$.

As explained above, for many of the physical problems, one couldn't expect that they always have classical solutions. In this course, we will also talk about the ideas on how to define weak solutions. Moreover, a very brief introduction on distributions will be given.

1.4. **Linear transport equations.** In the end of introduction, we use linear transport equation as an example to give a first try to get solution representation. It is actually an application of the solution for first order ordinary differential equation.

1.4.1. *Constant speed.* Linear transport equation in 1-D is the simplest partial differential equation,

$$\begin{aligned} \rho_t + a\rho_x &= 0, \qquad \text{in } \mathbb{R}\times\mathbb{R}_+, \\ \rho(x,0) &= \rho_0(x), \end{aligned}$$

where a is a constant. This equation with constant speed is an ordinary differential system in the sense that

$$\begin{aligned} &\frac{d}{dt}\rho(x(t),t) = 0, \\ &\frac{d}{dt}x(t) = a, \\ &\rho(x(0),0) = \rho_0(x_0). \end{aligned}$$

Obviously, it has solution

$$\rho(x,t) = \rho_0(x-at).$$

This strategy is called the **method of characteristics** and $x(t)$ is called the characteristic line.

Moreover, one can also solve the transport equation in inhomogeneous case

$$\begin{aligned} \rho_t + a\rho_x = f(x,t), &\qquad \text{in } \mathbb{R}\times\mathbb{R}_+, \\ \rho(x,0) = \rho_0(x) & \end{aligned}$$

by using the same characteristic line $x(t) = x_0 + at$. The solution is

$$\rho(x,t) = \rho_0(x-at) + \int_0^t f(x - a(t-s), s)ds.$$

Remark 1.1. By using the method of characteristics, one can easily solve the multi-D equation

$$\begin{aligned} \rho_t + \mathbf{b}\cdot\nabla\rho = 0, &\qquad \text{in } \mathbb{R}^n\times\mathbb{R}_+, \\ \rho(x,0) = \rho_0(x), & \end{aligned}$$

where $\mathbf{b}$ is a constant vector. We leave it as an exercise.

1.4.2. *Non-constant speed.* The Cauchy problem we consider in this part is

$$\begin{aligned} \rho_t + (v(x)\rho)_x = 0, &\qquad \text{in } \mathbb{R}\times\mathbb{R}_+, \\ \rho(x,0) = \rho_0(x), & \end{aligned} \tag{1.6}$$

where $v(x)$ is a given Lipschitz continuous function.

A reformulation of the equation is

$$\rho_t + v(x)\rho_x + v'(x)\rho = 0.$$

If $v(x)$ is Lipschitz continuous, then the characteristic line $x(t)$ satisfies

$$\begin{aligned} \frac{dx}{dt} &= v(x), \\ x(0) &= x_0. \end{aligned} \tag{1.7}$$

With the help of this line, the equation is rewritten into

$$\begin{aligned} \frac{d\rho(x(t;x_0),t)}{dt} &= -v'(x(t;x_0))\rho, \\ \rho(x(t;x_0),t)|_{t=0} &= \rho_0(x_0). \end{aligned}$$

By separation of variables in solving ODE, we have

$$\begin{aligned} \ln\rho(x(t;x_0),t) = \ln\rho_0(x_0) + \int_0^t -v'(x(\tau;x_0))d\tau & \\ = \ln\rho_0(x_0) + \int_{x_0}^{x(t;x_0)} -\frac{v'(x)}{v(x)}dx & \\ = \ln\rho_0(x_0) - \ln v(x(t;x_0)) + \ln v(x_0), & \end{aligned}$$

where in the first equation the change of variable $x = x(\tau, x_0)$, $dx = x' d\tau = v(x) d\tau$ is used. So the problem has solution

$$\rho(x(t; x_0), t) = \rho_0(x_0) \frac{v(x_0)}{v(x(t; x_0))}.$$

Therefore, finding the exact solution of the homogeneous problem (1.6) is reduced to the solvability of the characteristic line (1.7).

1.5. **Half-line problem.** We will study the half-line problem for transport equation only with constant speed. One must be careful in giving boundary conditions because of the "directions" of the characteristic lines. One can not arbitrarily give boundary condition at $x = 0$ if the characteristic lines starting from $t = 0$ meet $x = 0$ at some time t. As an example, we consider

$$\begin{aligned} u_t + u_x &= 0, \qquad (x, t) \in (0, +\infty) \times (0, +\infty), \\ u|_{t=0} &= u_0(x), \qquad u|_{x=0} = 0. \end{aligned}$$

For compatibility, we need $u_0(0) = 0$. It is easily seen that the solution is

$$u(x, t) = \begin{cases} 0, & x \le t, \\ u_0(x - t), & x > t. \end{cases}$$

If the boundary condition is inhomogeneous, say $u|_{x=0} = g(t)$, the compatibility condition is then $u(0, 0) = g(0)$. The solution changes accordingly to

$$u(x, t) = \begin{cases} g(t - x), & x \le t, \\ u_0(x - t), & x > t. \end{cases}$$

Another method is to use the transformation $v(x, t) = u(x, t) - g(t)$, so that v is zero on the boundary $x = 0$. And the problem that v satisfies is

$$\begin{aligned} v_t + v_x &= -g_t(t), \qquad (x, t) \in (0, +\infty) \times (0, +\infty), \\ v|_{t=0} &= u_0(x) - g(0), \qquad u|_{x=0} = 0. \end{aligned}$$

1.6. **Problems.**

(1) Point out the type of these equations (linear, semilinear, quasilinear, fully nonlinear).
  (a) $u_t - u_x u_{xxx} = x^2$,
  (b) $-\Delta u + u^2 = 1$,
  (c) $u_{tt} - \operatorname{div}((x^2 + t)\nabla u) = f(x)$,
  (d) $u_t + \operatorname{div}\Big[u \nabla\Big(\frac{\Delta\sqrt{u}}{\sqrt{u}}\Big)\Big] = 0$,
  (e) $\Delta(u^2) = f(x)$,
  (f) $|\nabla u| = 1$.

(2) Find the solution formula by using characteristic method.
  (a) $\begin{cases} u_t + (1 + x^2) u_x - u = 0, & t > 0, -\infty < x < \infty, \\ u|_{t=0} = \arctan x, & -\infty < x < \infty, \end{cases}$

(b) $$\begin{cases} u_t + \mathbf{b} \cdot \nabla u = f, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u|_{t=0} = g, & x \in \mathbb{R}^n. \end{cases}$$

2. Wave equation

After a short derivation of the one dimensional wave equation, we will give the solution formula for initial value problem (the Cauchy problem) by using characteristic method. Then with the help of spherical mean, we will derive the three dimensional Kirchhoff's formula and two dimensional Poisson's formula by Hadamard's method of descent. These shows that if the initial data are regular enough, then the classical solutions exist. Furthermore, the uniqueness and stability results will be given by using enery method.

For one dimensional initial boundary value problem with Dirichlet boundary condition, we will use the separation of variable to deduce a solution formula with the help of Fourier series. In the mean while, the Sturm Liouville theorem is going to be presented which provides the theoretical basis for the method of separation of variables. Based on that, the existence of solution can be obtained by studying the convergence of function series. Again energy method will be used to show uniqueness and stability.

In the end of this section, we introduce the weak solution of initial boundary value problem in the sense of distribution and show that the weak solution exists uniquely.

2.1. **Derivation of the one dimensional wave equation.** A stretched string with constant density ρ is set to vibrate on a plane. The displacement of the string $u(x,t)$ is a function of x and time t, where x is the horizontal coordinate. Now let's consider a discretized model in which a sequence of equally massed (i.e., $m=\rho h$) particles with equal distance h connect each other with coordinates $\cdots, x_{n-1}, x_n, x_{n+1}, \cdots$. Newton's second law implies that for the n-th particle, the external force on this particle is equal to its mass multiplied by the acceleration of it. We further make a simple assumption that the external force F on the n-th particle, which comes from the $(n-1)$-th and the $(n+1)$-th particle, is proportional to the difference of the displacements of its adjacent particles and itself, i.e.

$$F = (u(x_{n+1},t) - u(x_n,t))/h + (u(x_{n-1},t) - u(x_n,t))/h.$$

Therefore

$$\rho h u''(x_n,t) = \frac{u(x_{n+1},t) + u(x_{n-1},t) - 2u(x_n,t)}{h}.$$

Now if we take the limit $h \to 0$ and come back to the continuous case, we arrive at the one dimensional wave equation

$$\rho u_{tt} = u_{xx}.$$

2.2. **Cauchy problem.**

2.2.1. *Solution formula and existence.* **d'Alembert's formula — 1-D** We will give the formal solution of Cauchy problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \quad \text{in } \mathbb{R}\times\mathbb{R}_+, \\ u|_{t=0} &= g(x), \\ u_t|_{t=0} &= h(x). \end{aligned} \tag{2.1}$$

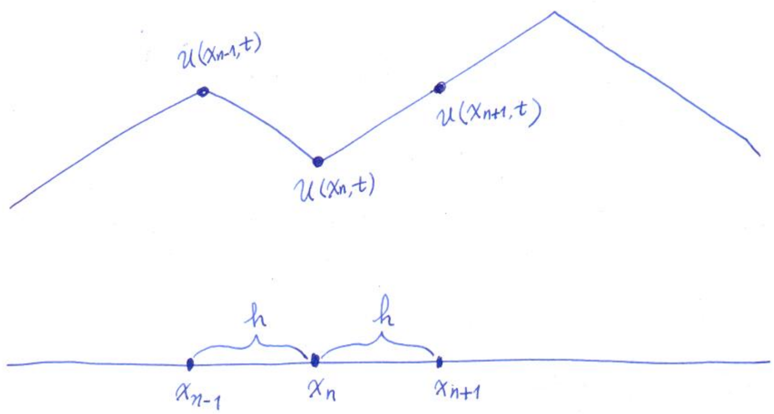

FIGURE 1. Derivation of wave equation by particle method

By factorizing the operator $\partial_{tt} - \partial_{xx} = (\partial_t + \partial_x)(\partial_t - \partial_x)$, the problem is reduced into solving the following two transport equations,

$$\begin{aligned} &v_t + v_x = 0, \\ &v|_{t=0} = h(x) - g'(x) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &u_t - u_x = v, \\ &u|_{t=0} = g(x). \end{aligned} \tag{2.3}$$

With the help of characteristic method, the solution of (2.2) is

$$v(x,t) = h(x-t) - g'(x-t)$$

and the solution of (2.3) is

$$u(x,t) = g(x+t) + \int_0^t v(x+(t-s), s)ds.$$

Therefore, combining the two formulas above, we have the solution of (2.1),

$$\begin{aligned} u(x,t) &= g(x+t) + \int_0^t h(x+t-2s) - g'(x+t-2s)ds \\ &= g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} (h(y) - g'(y))dy \\ &= g(x+t) - \frac{1}{2}g(x+t) + \frac{1}{2}g(x-t) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy \\ &= \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy. \end{aligned}$$

This is the so-called **d'Alembert's formula**. The formal solution of (2.1) is

$$u(x,t)=\frac{1}{2}(g(x+t)+g(x-t))+\frac{1}{2}\int_{x-t}^{x+t}h(y)dy. \tag{2.4}$$

From this formula, we are ready to get the existence of the solution with smooth initial data

Theorem 2.1. *If $g\in C^2(\mathbb{R})$ and $h\in C^1(\mathbb{R})$, then u, given by d'Alembert's furmula, is a $C^2(\mathbb{R}\times[0,+\infty))$-function. Furthermore, it satisfies wave equation $u_{tt}-u_{xx}=0$ and*

$$\lim_{(x,t)\to(x_0,0)}u(x,t)=g(x_0),\qquad \lim_{(x,t)\to(x_0,0)}u_t(x,t)=h(x_0).$$

Some properties of the solution d'Alembert's formula reveals in itself some important sets in the $x-t$ space.

(1) $\{y\in\mathbb{R}||y-x|\le t\}$ is called the **domain of dependence** of the point (x,t),
(2) $\{(x,t)\in\mathbb{R}\times[0,+\infty)|x\ge x_1-t \text{ and } x\le x_2+t\}$ is called the **range of influence** of $[x_1,x_2]$,
(3) $\{(x,t)\in\mathbb{R}\times[0,+\infty)|x\ge x_1+t \text{ and } x\le x_2-t\}$ is called the **determining region** of $[x_1,x_2]$,
(4) $x+t$ and $x-t$ are the **characteristics** of the wave equation.

Inhomogeneous problem By the same method, we can also find the solution of inhomogeneous problem

$$\begin{aligned} u_{tt}-u_{xx}&=f(x,t),\quad (x,t)\in\mathbb{R}\times\mathbb{R}_+, \\ u|_{t=0}&=g(x),\qquad u_t|_{t=0}=h(x), \end{aligned} \tag{2.5}$$

which is

$$u(x,t)=\frac{1}{2}(g(x+t)+g(x-t))+\frac{1}{2}\int_{x-t}^{x+t}h(y)dy+\frac{1}{2}\int_0^t ds\int_{x-(t-s)}^{x+(t-s)}f(y,s)dy. \tag{2.6}$$

By using the solution formula, it is easy to verify that

Corollary 2.1. *If g, h and f are odd (even, or periodic) in x, so is u.*

Half-line problem By using extension, we can also give the solution formula of the following problem

$$\begin{aligned} u_{tt}-u_{xx}&=0,\qquad \text{in } \mathbb{R}_+\times\mathbb{R}_+, \\ u|_{t=0}&=g(x),\quad u_t|_{t=0}=h(x), \\ u|_{x=0}&=0. \end{aligned} \tag{2.7}$$

In order to assure compatibility between initial and boundary conditions, we need $h(0)=g(0)=0$. The homogeneous Dirichlet boundary condition at $x=0$ motivates us to use odd extension. Let

$$\tilde{g}=\begin{cases} g(x), & x\ge 0, \\ -g(-x), & x<0, \end{cases}$$

and we do the same odd extensions for $\tilde{u}(x,t)$ and $\tilde{h}(x)$. Then from the above corollary, $\tilde{u}$ satisfies the following Cauchy problem

$$\begin{aligned}\tilde{u}_{tt} - \tilde{u}_{xx} &= 0, \quad \text{in } \mathbb{R} \times \mathbb{R}_+,\\ \tilde{u}|_{t=0} &= \tilde{g}(x),\\ \tilde{u}_t|_{t=0} &= \tilde{h}(x).\end{aligned}$$

By d'Alembert's formula, $\tilde{u}$ has the representation

$$\tilde{u}(x,t) = \frac{1}{2}(\tilde{g}(x+t) + \tilde{g}(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)dy.$$

We need to get back to the domain $\{(x,t) : x > 0, t > 0\}$ and drop the tildes in the formula. There are two cases. In the case of $x \geq t$, our solution has the representation

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy.$$

In the case of $0 \leq x < t$, the solution is given by

$$\begin{aligned}u(x,t) &= \frac{1}{2}(g(x+t) - g(t-x)) + \frac{1}{2}\Big(\int_0^{x+t} h(y)dy - \int_0^{t-x} h(-y)dy\Big)\\ &= \frac{1}{2}(g(x+t) - g(t-x)) + \frac{1}{2}\int_{t-x}^{x+t} h(y)dy.\end{aligned}$$

Remark 2.1. For inhomogeneous boundary condition $u|_{x=0} = u_D(t)$, we can first get a homogeneous boundary condition by using new variable $v = u - u_D(t)$, where v satisfies $v_{tt} - v_{xx} = -(u_D)_{tt}$. Afterwards, we do the same odd extension to get the solution formula.

Remark 2.2. It is an easy exercise to get half-line problem with homogeneous Neumann boundary condition $u_x|_{x=0} = 0$ by using even extension.

Kirchhoff's formula in 3-D and Poisson's formula in 2-D We will reduce the multi-dimension problem into a half-line problem by using **spherical mean** of the solution.

The spherical mean of a function $u(x,t)$ on $\partial B(x,r)$ is given by

$$U(x;r,t) = \fint_{\partial B(x,r)} u(y,t)dS_y. \tag{2.8}$$

Lemma 2.1. *If $u \in C^m(\mathbb{R}^n \times (0,+\infty))$ is a solution of*

$$\begin{aligned}u_{tt} - \Delta u &= 0, \quad \text{in } \mathbb{R}^n \times (0,+\infty),\\ u|_{t=0} &= g, \qquad u_t|_{t=0} = h.\end{aligned} \tag{2.9}$$

Then the spherical mean of u, $U(x;r,t)$ as a function of r and t is a function in $C^m([0,+\infty)\times [0,+\infty))$. Furthermore it satisfies

$$\begin{aligned}U_{tt} - U_{rr} - \frac{n-1}{r}U_r &= 0, \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+,\\ U|_{t=0} = G, &\quad U_t|_{t=0} = H,\end{aligned} \tag{2.10}$$

where G, H are the corresponding spherical mean of g, h. The equation (2.10) *is called the* **Euler-Poisson-Darboux equation***.*

Proof. By direct calculations, we have

$$\begin{aligned} U_r(x;r,t) &= \frac{\partial}{\partial r} \fint_{\partial B(x,r)} u(y,t)dS_y = \frac{\partial}{\partial r} \fint_{\partial B(0,1)} u(x+rz,t)dS_z \\ &= \fint_{\partial B(0,1)} \nabla u(x+rz,t)\cdot z dS_z = \fint_{\partial B(x,r)} \nabla u(y,t)\cdot \frac{y-x}{r} dS_y \\ &= \fint_{\partial B(x,r)} \nabla u \cdot \gamma dS_y = \frac{r}{n} \fint_{B(x,r)} \Delta u(y,t)dy. \end{aligned}$$

As a consequence,

$$\lim_{r\to 0+} U_r(x;r,t) = 0.$$

If we take one derivative more,

$$\begin{aligned} U_{rr}(x;r,t) &= \frac{\partial}{\partial r}\Big(\frac{r}{n}\fint_{B(x,r)} \Delta u(y,t)dy\Big) = \frac{1}{n\alpha(n)}\frac{\partial}{\partial r}\Big(r^{1-n}\int_{B(x,r)} \Delta u(y,t)dy\Big) \\ &= \frac{1-n}{n}\frac{1}{\alpha(n)r^n}\int_{B(x,r)} \Delta u(y,t)dy + \frac{1}{n\alpha(n)r^{n-1}}\frac{\partial}{\partial r}\int_{B(x,r)} \Delta u(y,t)dy \\ &= (\frac{1}{n}-1)\fint_{B(x,r)} \Delta u dy + \fint_{\partial B(x,r)} \Delta u dS_y. \end{aligned}$$

where $\alpha(n) = \pi^{\frac{n}{2}}/\Gamma(\frac{n}{2}+1)$ is the volume of the n dimensional unit ball. Furthermore,

$$\lim_{r\to 0+} U_{rr}(x;r,t) = \frac{1}{n}\Delta u(x,t).$$

Therefore, if $u \in C^2$, we have $U \in C^2$. By iteration argument, we have $U \in C^m$ if $u \in C^m$.

Back to the first order derivative, by using the wave equation $u_{tt} - \Delta u = 0$, we have

$$U_r = \frac{r}{n}\fint_{B(x,r)} u_{tt}dy = \frac{1}{n\alpha(n)r^{n-1}}\int_{B(x,r)} u_{tt}dy.$$

Multiplying the above equation by r^{n-1} gives

$$r^{n-1}U_r = \frac{1}{n\alpha(n)}\int_{B(x,r)} u_{tt}dy.$$

Then the desired equation follows from taking one more derivative of $r^{n-1}U_r$, i.e.,

$$(r^{n-1}U_r)_r = \frac{1}{n\alpha(n)}\int_{\partial B(x,r)} u_{tt}dS_y = r^{n-1}\fint_{\partial B(x,r)} u_{tt}dS_y = r^{n-1}U_{tt}.$$

□

In the case of $n = 3$, we will get Kirchhoff's formula by using Euler-Poisson-Darboux equation.

Let $\widetilde{U} = rU$, $\widetilde{G} = rG$ and $\widetilde{H} = rH$, we have

$$\widetilde{U}_r = U + rU_r,$$

and moreover

$$\widetilde{U}_{tt} = rU_{tt} = rU_{rr} + 2U_r = (U + rU_r)_r = \widetilde{U}_{rr}.$$

Now $\widetilde{U}$ solves the half-line problem

$$\begin{aligned} \widetilde{U}_{tt} - \widetilde{U}_{rr} = 0, &\quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+, \\ \widetilde{U}|_{t=0} = \widetilde{G}, &\quad \widetilde{U}_t|_{t=0} = \widetilde{H}, \\ \widetilde{U}|_{r=0} = 0. & \end{aligned}$$

By the solution representation in half-line problem, we have

$$\widetilde{U}(x; r, t) = \frac{1}{2}(\widetilde{G}(r+t) - \widetilde{G}(t-r)) + \frac{1}{2}\int_{t-r}^{r+t} \widetilde{H}(y)dy, \quad \forall 0 < r < t.$$

Since $u(x,t)$ is a continuous function, its value at (x,t) is exactly the limit of its spherical mean. Thus

$$\begin{aligned} u(x,t) &= \lim_{r\to 0^+} U(x;r,t) = \lim_{r\to 0^+} \frac{\widetilde{U}(x;r,t)}{r} \\ &= \lim_{r\to 0^+} \left[\frac{1}{2}\frac{\widetilde{G}(r+t) - \widetilde{G}(t-r)}{r} + \frac{1}{2r}\int_{t-r}^{r+t} \widetilde{H}(y)dy\right] \\ &= \widetilde{G}'(t) + \widetilde{H}(t). \end{aligned}$$

By the definition of $\widetilde{G}$ and $\widetilde{H}$, after changing back to the original functions g and h, we arrive at

$$u(x,t) = \frac{\partial}{\partial t}\Big[t \fint_{\partial B(x,t)} g(y)dS_y\Big] + t\fint_{\partial B(x,t)} h(y)dS_y.$$

Therefore after the following further computation

$$\begin{aligned} &\frac{\partial}{\partial t}\fint_{\partial B(x,t)} g(y)dS_y = \frac{\partial}{\partial t}\fint_{\partial B(0,1)} g(x+tz)dS_z \\ = &\fint_{\partial B(0,1)} \nabla g(x+tz)\cdot z dS_z = \fint_{\partial B(x,t)} \nabla g(y)\cdot \frac{y-x}{t}dS_y, \end{aligned}$$

we obtain the 3-D **Kirchhoff's formula**,

$$u(x,t) = \fint_{\partial B(x,t)} [g(y) + \nabla g(y)\cdot(y-x) + th(y)]dS_y. \tag{2.11}$$

In the case of $n = 2$, we will get Poisson's formula by the Hadamard's method of descent.

If $u(x_1, x_2, t)$ is a solution in 2-D, let $\bar{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t)$, then $\bar{u}$ solves the wave equation in 3-D,

$$\begin{aligned} \bar{u}_{tt} - \Delta\bar{u} = 0, &\quad \text{in } \mathbb{R}^3 \times (0,\infty), \\ \bar{u}|_{t=0} = \bar{g}, &\quad \bar{u}_t|_{t=0} = \bar{h}, \end{aligned}$$

where $\bar{g}(x_1,x_2,x_3) = g(x_1,x_2)$ and $\bar{h}(x_1,x_2,x_3) = h(x_1,x_2)$. Then by Kirchhoff's formula in 3-D, we have

$$u(x,t) = \frac{\partial}{\partial t}\Big[t\fint_{\partial B(\bar{x},t)} \bar{g}(y)dS_y\Big] + t\fint_{\partial B(\bar{x},t)} \bar{h}(y)dS_y,$$

where $\bar{x} = (x_1, x_2, 0)$. Due to the fact that $\bar{g}(y_1, y_2, y_3) = g(y_1, y_2)$, we can simplify the integral on $\partial B(\bar{x}, t)$ by

$$\begin{aligned} \fint_{\partial B(\bar{x},t)} \bar{g}(y) dS_y &= \frac{1}{4\pi t^2} \int_{\partial B(\bar{x},t)} \bar{g}(y) dS_y \\ &= \frac{2}{4\pi t^2} \int_{B(x,t)} g(y)(1 + |\nabla\gamma(y)|^2)^{\frac{1}{2}} dy, \end{aligned}$$

where $\gamma(y) = \sqrt{t^2 - (y-x)^2}$ and $(1 + |\nabla\gamma(y)|^2)^{\frac{1}{2}} = t(t^2 - |y-x|^2)^{-\frac{1}{2}}$. Therefore

$$\begin{aligned} \fint_{\partial B(\bar{x},t)} \bar{g}(y) dS_y &= \frac{1}{2\pi t} \int_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{\frac{1}{2}}} dy \\ &= \frac{t}{2} \fint_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{\frac{1}{2}}} dy. \end{aligned}$$

Then after taking derivative with respect to t, we have

$$\begin{aligned} & \frac{\partial}{\partial t}\left[t^2 \fint_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{\frac{1}{2}}} dy \right] = \frac{\partial}{\partial t}\left[t \fint_{B(0,1)} \frac{g(x+tz)}{(1-|z|^2)^{\frac{1}{2}}} dz \right] \\ = & \fint_{B(0,1)} \frac{g(x+tz)}{(1-|z|^2)^{\frac{1}{2}}} dz + t \fint_{B(0,1)} \frac{\nabla g(x+tz) \cdot z}{(1-|z|^2)^{\frac{1}{2}}} dz \\ = & \; t \fint_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{\frac{1}{2}}} dy + t \fint_{B(x,t)} \frac{\nabla g(y) \cdot (y-x)}{(t^2 - |y-x|^2)^{\frac{1}{2}}} dy \end{aligned}$$

Thus the 2-D **Poisson's formula** is

$$u(x,t) = \frac{1}{2} \fint_{B(x,t)} \frac{tg(y) + t^2 h(y) + t\nabla g(y) \cdot (y-x)}{(t^2 - |y-x|^2)^{\frac{1}{2}}} dy. \tag{2.12}$$

With the help of Kirchhoff's and Poisson's formula in 3-D and 2-D, we have the following existence result.

Theorem 2.2. *($n = 2, 3$) If $g \in C^3(\mathbb{R}^n)$ and $h \in C^2(\mathbb{R}^n)$, then u given by* (2.11) *and* (2.12)*, a function in $C^2(\mathbb{R}^n \times [0, +\infty))$, is a classical solution of* (2.9)*.*

The difference of the solution behaviors between 3-D and 2-D If we look at the Kirchhoff's formula (2.11) and the Poisson's formula (2.12), we can easily find that the main difference locates on the domain of the integrals. The integral is on the sphere (which is the boundary of a domain) in Kirchhoff's formula, while the integral is in the ball in Poisson's formula.

Let's assume that the initial data have compact support Ω, where Ω is connected and regular enough. For any fixed point $x_0 \notin \Omega$, let $d_1 = \text{dist}(x_0, \Omega) > 0$, $d_2 = \max\{\text{dist}(x_0, x) : x \in \Omega\}$. In 3-D case, the possible non-zero points of $u(x_0, t)$ can only be in the interval $t \in [d_1, d_2]$, while in 2-D, the possible non-zero points of $u(x_0, t)$ must be the half line $t \in [d_1, +\infty)$. That explains why one can hear the others' voice in 3-D, and why the water wave diffuses to the whole space in 2-D.

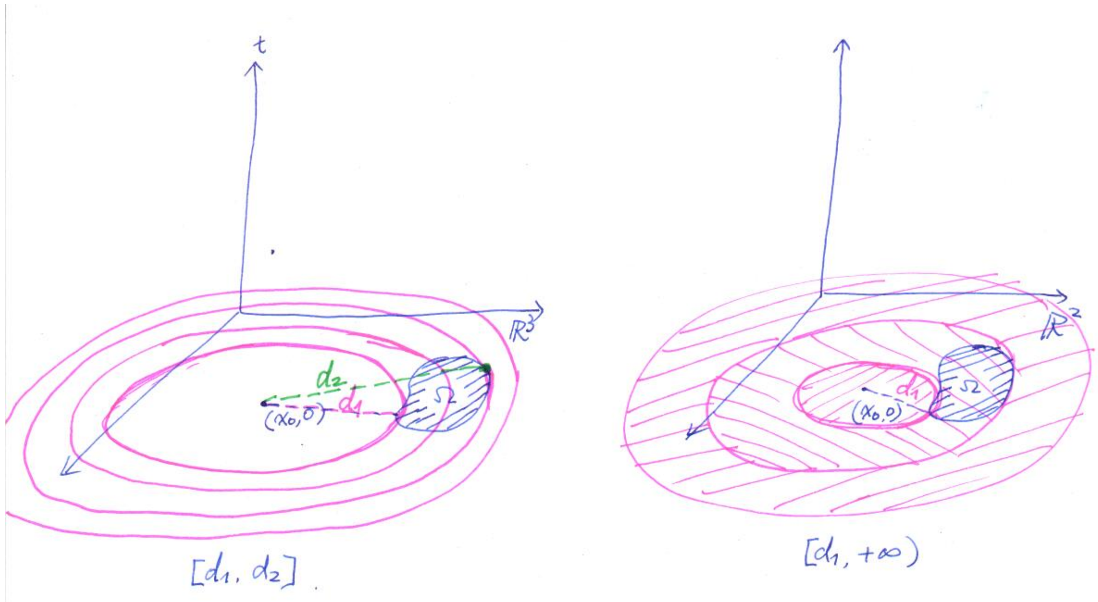

FIGURE 2. Wave propagation in 3-D and 2-D

2.2.2. *Uniqueness and stability (Energy method).* After obtaining the existence result, a further natural question to ask is whether the solution is unique and stable? In the following discussion, we introduce the energy method, which is a "powerful" tool in PDE theory. Uniqueness and stability follow thereafter as an application of the energy estimates.

We first introduce a useful lemma.

Lemma 2.2. *(Gronwall's inequality) Assume $G(\tau) \geq 0$, $G'(\tau) \in C[0,T]$ and $\forall \tau \in [0,T]$, the following inequality holds*

$$\frac{dG(\tau)}{d\tau} \leq CG(\tau) + F(\tau),$$

where C is a nonnegative constant, $F(\tau) \geq 0$ is nondecreasing in τ. Then

$$\frac{dG(\tau)}{d\tau} \leq Ce^{C\tau}G(0) + e^{C\tau}F(\tau),$$

and

$$G(\tau) \leq e^{C\tau}G(0) + C^{-1}(e^{C\tau} - 1)F(\tau).$$

Proof. By multiplying the given inequality by $e^{-C\tau}$ and integrating it over $[0,\tau]$, we have

$$e^{-C\tau}G(\tau) \leq G(0) + \int_0^\tau e^{-Ct}F(t)dt \leq G(0) + F(\tau)C^{-1}(1 - e^{-C\tau}).$$

$\square$

The Cauchy problem we considered is revisited

$$\begin{aligned} u_{tt} - u_{xx} &= f, \quad \text{in } \mathbb{R} \times \mathbb{R}_+, \\ u|_{t=0} &= g(x), \\ u_t|_{t=0} &= h(x). \end{aligned} \tag{2.13}$$

The energy inequality of 1-D Cauchy problem (2.13) is

Theorem 2.3. *If $u \in C^1(\mathbb{R} \times [0, +\infty)) \cap C^2(\mathbb{R} \times (0, +\infty))$ is a solution of (2.13), then $\forall (x_0, t_0) \in \mathbb{R} \times (0, +\infty)$, we have*

$$\int_{\Omega_\tau} [u_t^2(x,\tau) + u_x^2(x,\tau)]dx \le C(t_0)\Big(\int_{\Omega_0} (h^2 + g_x^2)dx + \int\int_{K_\tau} f^2(x,t)dxdt\Big),$$

$$\int\int_{K_\tau} [u_t^2(x,t) + u_x^2(x,t)]dxdt \le C(t_0)\Big(\int_{\Omega_0} (h^2 + g_x^2)dx + \int\int_{K_\tau} f^2(x,t)dxdt\Big),$$

where $K = \{(x,t) \in \mathbb{R} \times [0, +\infty) : |x - x_0| < t_0 - t\}$, $K_\tau = K \cap \{0 \le t \le \tau\}$, $\Omega_\tau = \bar{K} \cap \{t = \tau\}$, and $C(t_0)$ is a constant depends on t_0.

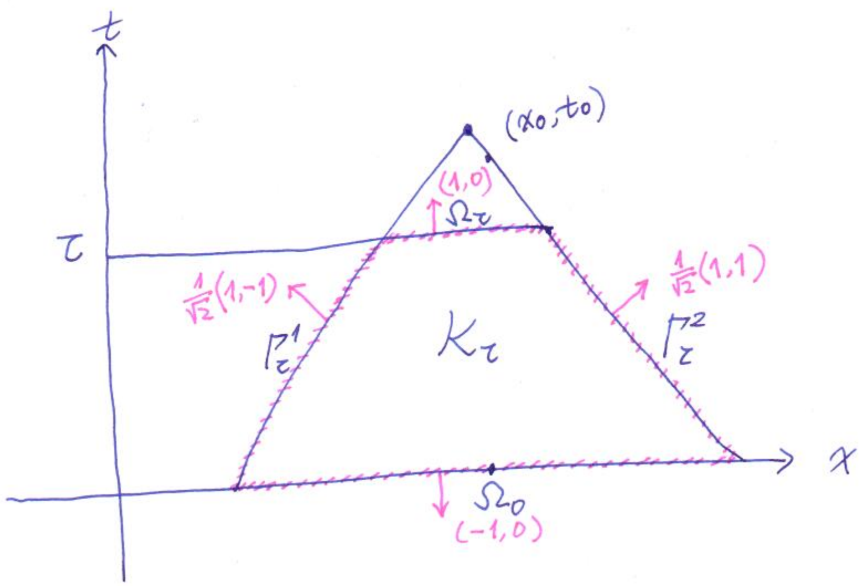

FIGURE 3. Domains in energy estimates

Proof. Multiplying the equation by u_t and integrating over K_τ, we have

$$\int\int_{K_\tau} (u_t u_{tt} - u_t u_{xx})dxdt = \int\int_{K_\tau} u_t f dxdt.$$

Noticing that the boundary of K_τ is $\partial K_\tau = \Omega_0 \cup \Omega_\tau \cup \Gamma_\tau^1 \cup \Gamma_\tau^2$, we can calculate the left hand side by using divergence lemma,

$$\begin{aligned}
& \int\int_{K_\tau} \frac{1}{2}(u_t^2+u_x^2)_t dxdt - \int\int_{K_\tau} (u_t u_x)_x dxdt \\
= & \int_{\partial K_\tau} (\frac{1}{2}(u_t^2+u_x^2), -u_t u_x)^T \cdot \gamma dl \\
= & \int_{\Omega_\tau} \frac{1}{2}(u_t^2+u_x^2)dx - \int_{\Omega_0} \frac{1}{2}(u_t^2+u_x^2)dx \\
& + \int_{\Gamma_\tau^1} \frac{1}{\sqrt{2}}(\frac{1}{2}(u_t^2+u_x^2)+u_t u_x)dl + \int_{\Gamma_\tau^2} \frac{1}{\sqrt{2}}(\frac{1}{2}(u_t^2+u_x^2)-u_t u_x)dl \\
\geq & \int_{\Omega_\tau} \frac{1}{2}(u_t^2+u_x^2)dx - \int_{\Omega_0} \frac{1}{2}(h^2+g_x^2)dx,
\end{aligned}$$

where γ is the outer unit normal vector of ∂K_τ and has values $\gamma = (-1, 0)$ on Ω_0, $\gamma = (1, 0)$ on Ω_τ, $\gamma = \frac{1}{\sqrt{2}}(1, -1)$ on Γ_τ^1 and $\gamma = \frac{1}{\sqrt{2}}(1, 1)$ on Γ_τ^2.

The right hand side can be estimated by using Young's inequality,

$$\int\int_{K_\tau} u_t f dxdt \leq \frac{1}{2}\int\int_{K_\tau} u_t^2 dxdt + \frac{1}{2}\int\int_{K_\tau} f^2 dxdt.$$

with the above discussions being combined together, we have

$$\int_{\Omega_\tau} (u_t^2+u_x^2)dx \leq \int_{\Omega_0} (h^2+g_x^2)dx + \int\int_{K_\tau} u_t^2 dxdt + \int\int_{K_\tau} f^2 dxdt.$$

Let

$$\begin{aligned}
G(\tau) &= \int\int_{K_\tau} (u_t^2+u_x^2)dxdt = \int_0^\tau \int_{x_0-(t_0-t)}^{x_0+(t_0-t)} (u_t^2+u_x^2)dxdt, \\
F(\tau) &= \int_{\Omega_0} (h^2+g_x^2)dx + \int\int_{K_\tau} f^2 dxdt.
\end{aligned}$$

The estimates we have done is equivalent to the following inequality

$$\frac{dG(\tau)}{d\tau} \leq G(\tau) + F(\tau),$$

where $F(\tau)$ is increasing in τ. Then noticing that $G(0) = 0$, Gronwall's inequality implies

$$G(\tau) \leq (e^\tau - 1)F(\tau) \leq e^{t_0} F(\tau).$$

□

We can also get the L^2 estimate from the energy estimate.

Theorem 2.4. *If $u \in C^1(\mathbb{R} \times [0,+\infty)) \cap C^2(\mathbb{R} \times (0,+\infty))$ is a solution of* (2.13)*, then* $\forall (x_0, t_0) \in \mathbb{R} \times (0,+\infty)$,

$$
\begin{aligned}
\int_{\Omega_\tau} u^2(x,\tau)dx &\leq M_1\Big(\int_{\Omega_0}(g^2+h^2+g_x^2)dx + \int\int_{K_\tau} f^2 dxdt\Big),\\
\int\int_{K_\tau} u^2(x,t)dxdt &\leq M_1\Big(\int_{\Omega_0}(g^2+h^2+g_x^2)dx + \int\int_{K_\tau} f^2 dxdt\Big),
\end{aligned}
$$

where M_1 is a constant depends on t_0, $\tau \in [0, t_0]$ and the definitions of domains K_τ, Ω_τ and Ω_0 are the same as before.

Proof. We only need to prove that $\|u\|_{L^2(\Omega_\tau)}$ and $\|u\|_{L^2(K_\tau)}$ can be controlled by $\|u_t\|_{L^2(K_\tau)}$. In fact,

$$
\int_{\Omega_\tau}(u^2(x,\tau)-u^2(x,0))dx = \int_{\Omega_\tau}\int_0^\tau \partial_t u^2(x,t)dtdx \leq \int\int_{K_\tau}(u^2+u_t^2)dxdt.
$$

By Gronwall's inequality, we have

$$
\begin{aligned}
\int_{\Omega_\tau} u^2(x,\tau)dx \leq C(t_0)\Big(\int_{\Omega_0} g^2(x)dx + \int\int_{K_\tau} u_t^2 dxdt\Big),\\
\int\int_{K_\tau} u^2(x,t)dxdt \leq C(t_0)\Big(\int_{\Omega_0} g^2(x)dx + \int\int_{K_\tau} u_t^2 dxdt\Big).
\end{aligned}
$$

Thus the L^2 estimate is a direct consequence from energy estimates. □

Uniqueness is a direct corollary of energy estimates. Let $Q = \mathbb{R} \times (0,+\infty)$.

Corollary 2.2. *If u_1 and u_2 are two $C^2(Q) \cap C^1(\bar{Q})$ solutions of the Cauchy problem* (2.13)*, then $u_1 = u_2$ in Q.*

Proof. Let $w = u_1 - u_2$, then $w_{tt} - w_{xx} = 0$ in Q and $w|_{t=0} = w_t|_{t=0} = 0$. Then energy estimates for w state that $\forall (x_0, t_0) \in Q$, $0 < \tau < t_0$, there holds

$$
\int_{\Omega_\tau}(w^2 + w_t^2 + w_x^2)dx \leq 0,
$$

which implies $w = 0$ in Ω_τ. Since (x_0, t_0) is arbitrary, the uniqueness is proved. □

Stability in the sense of H^1-norm. H^1-norm of a function is defined by

$$
\|u\|_{H^1} = \|u\|_{L^2} + \|\nabla u\|_{L^2}.
$$

Corollary 2.3. *If u_1, u_2 are $C^2(Q) \cap C^1(\bar{Q})$ solutions of the Cauchy problem* (2.13) *with different data f_1, g_1, h_1 and f_2, g_2, h_2 respectively. Then*

$$
\|u_1 - u_2\|_{H^1(K_\tau)} \leq M\left(\|g_1 - g_2\|_{H^1(\Omega_0)} + \|h_1 - h_2\|_{L^2(\Omega_0)} + \|f_1 - f_2\|_{L^2(K_\tau)}\right),
$$

where

$$\begin{aligned}
\|u\|^2_{H^1(K_\tau)} &= \int\int_{K_\tau} [u^2(x,\tau) + u_t^2(x,\tau) + u_x^2(x,\tau)]dxdt,\\
\|g\|^2_{H^1(\Omega_0)} &= \int_{\Omega_0} [g^2(x) + g_x^2(x)]dx,\\
\|h\|^2_{L^2(\Omega_0)} &= \int_{\Omega_0} h^2(x)dx,\\
\|f\|^2_{L^2(K_\tau)} &= \int\int_{K_\tau} f^2(x,t)dxdt,
\end{aligned}$$

and the definitions of domains K_τ, Ω_τ and Ω_0 are the same as before.

Proof. Notice that $w = u_1 - u_2$ is a solution of

$$\begin{aligned}
w_{tt} - w_{xx} = f_1 - f_2, &\quad \text{in } Q,\\
w|_{t=0} = g_1 - g_2, &\quad w_t|_{t=0} = h_1 - h_2,
\end{aligned}$$

then the energy estimates and L^2-estimates directly imply the stability. □

2.3. **Initial boundary value problem in one dimension.** We consider the initial boundary value problem with homogeneous Dirichlet boundary condition

$$\begin{aligned}
u_{tt} - u_{xx} = 0, &\qquad x \in (0,1), t > 0,\\
u|_{t=0} = g(x), &\qquad u_t|_{t=0} = h(x),\\
u|_{x=0} = u|_{x=1} = 0. &
\end{aligned} \tag{2.14}$$

Other reasonable boundary conditions are Neumann boundary condition and Robin boundary condition. For simplicity, in most of the cases, we only handle the Dirichlet boundary condition.

2.3.1. *Separation of variables (motivation).* The problem (2.14) can be solved by separation of variables. Before stating the rigorous result, we present here a formal calculation as a motivation.

Suppose that our solution has a factorized form $u(x,t) = X(x)T(t)$. If we put it into the equation, we get immediately

$$XT'' - X''T = 0 \quad \Rightarrow \quad \frac{X''}{X} = \frac{T''}{T}.$$

Now both sides of this equation are functions of one dimensional variables, but with different variables x and t. Once they are equal, they must be the same constant, which is independent of x and t. We denote the constant by $-\lambda$. By applying the boundary conditions, we have

$$\begin{aligned}
X'' + \lambda X = 0, &\qquad X(0) = X(1) = 0,\\
T'' + \lambda T = 0. &
\end{aligned}$$

Then the solutions would be

$$X(x) = C\cos\sqrt{\lambda}x + D\sin\sqrt{\lambda}x,$$
$$T(t) = A\cos\sqrt{\lambda}t + B\sin\sqrt{\lambda}t,$$

where A, B, C and D are constants to be determined later. The reason why λ are nonnegative will be explained later.

Applying the boundary condition for X yields

$$C = 0, \quad D\sin\sqrt{\lambda} = 0, \quad \Rightarrow \quad \lambda = (n\pi)^2, \quad n = 1, 2, 3, \cdots.$$

Therefore, for any fixed n, we have the following solutions with undetermined constants A_n and B_n,

$$\begin{aligned} u_n(x,t) &= (A_n\cos\sqrt{\lambda_n}t + B_n\sin\sqrt{\lambda_n}t)\sin\sqrt{\lambda_n}x \\ &= (A_n\cos n\pi t + B_n\sin n\pi t)\sin n\pi x. \end{aligned}$$

By superposition principle, we know that the finite summation of solutions is still a solution, i.e., for any fixed N,

$$u_N(x,t) = \sum_{n=1}^{N} u_n(x,t) = \sum_{n=1}^{N}(A_n\cos n\pi t + B_n\sin n\pi t)\sin n\pi x$$

is a solution of the wave equation with Dirichlet boundary condition. If, furthermore, the initial data $g(x)$ and $h(x)$ have the same form, say,

$$g(x) = \sum_{n=1}^{N} g_n\sin n\pi x, \quad h(x) = \sum_{n=1}^{N} h_n\sin n\pi x,$$

then u_N must be a solution with this initial data. One could naturally ask how about the case with general initial data? What is the corresponding solution $u(x,t)$? Can we use ∞ to replace N? These problems introduce directly to the theory of Fourier series. We first write initial data g and h into Fourier series, which can be done for L^2 functions (see the appendix in the end of this chapter), then the corresponding series $\lim_{N\to\infty} u_N(x,t)$ can be expected to be a solution for smooth enough initial data. We will prove this rigorously later.

2.3.2. *Generalized Fourier series (Sturm-Liouville theorem).* As a preparation for getting rigorous solution formula, we introduce the so-called Sturm-Liouville problem, which is the theoretical basis of the method of separation of variables. We consider an eigenvalue problem with more general boundary condition (Robin type). Dirichlet and Neumann boundary conditions are merely special cases from Robin boundary condition. Now we consider

$$\begin{aligned} X'' + \lambda X = 0, &\quad 0 < x < 1, \\ -\alpha_1 X'(0) + \beta_1 X(0) = 0, &\quad \alpha_i, \beta_i \geq 0, \\ \alpha_2 X'(1) + \beta_2 X(1) = 0, &\quad \alpha_i + \beta_i > 0. \end{aligned} \tag{2.15}$$

Theorem 2.5. *(Sturm-Liouville theorem)*

(1) *All the eigenvalues of* (2.15) *are nonnegative. In addition, if* $\beta_1 + \beta_2 > 0$*, then all the eigenvalues are positive,*

(2) *eigenvalues are countable and increasing to infinity, i.e.,*

$$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \quad \lim_{n\to\infty} \lambda_n = \infty,$$

(3) *eigenfunctions of different eigenvalues are orthogonal in the following sense*

$$\int_0^1 X_\lambda X_\mu dx = 0, \quad \text{for } \lambda \neq \mu,$$

(4) $\forall f \in L^2(0,1)$*, it holds that*

$$f(x) = \sum_{n=1}^{\infty} C_n X_n(x), \text{ in the sense of } L^2(0,1), \quad C_n = \frac{\int_0^1 f(x) X_n(x) dx}{\int_0^1 X_n^2 dx},$$

or

$$\lim_{n\to\infty} \|f(x) - f_n(x)\|_{L^2} = 0,$$

where $\sum_{n=1}^{\infty} C_n X_n(x)$ *is called the generalized Fourier series.*

Proof. We will only proof the first three statements. The last one can be found in functional analysis in the part of compact self-adjoint operators.

(1) By multiplying the equation by X_λ and integrating it over $(0,1)$, we have

$$X_\lambda X_\lambda' \Big|_0^1 - \int_0^1 (X_\lambda')^2 dx + \lambda \int_0^1 X_\lambda^2 dx = 0.$$

Boundary conditions show that

$$\begin{aligned} -\alpha_1 X_\lambda'(0) X_\lambda(0) + \beta_1 X_\lambda^2(0) = 0, &\qquad -\alpha_1 (X_\lambda'(0))^2 + \beta_1 X_\lambda(0) X_\lambda'(0) = 0, \\ \alpha_2 X_\lambda'(1) X_\lambda(1) + \beta_2 X_\lambda^2(1) = 0, &\qquad \alpha_2 (X_\lambda'(1))^2 + \beta_2 X_\lambda(1) X_\lambda'(1) = 0. \end{aligned}$$

From these, we get

$$\begin{aligned} X_\lambda'(0) X_\lambda(0) &= \frac{1}{\alpha_1 + \beta_1} (\alpha_1 (X_\lambda'(0))^2 + \beta_1 X_\lambda^2(0)), \\ X_\lambda'(1) X_\lambda(1) &= \frac{-1}{\alpha_2 + \beta_2} (\beta_2 X_\lambda^2(1) + \alpha_2 (X_\lambda'(1))^2). \end{aligned}$$

As a consequence, we know the nonnegativity of

$$\lambda \int_0^1 X_\lambda^2 dx = \int_0^1 (X_\lambda')^2 dx - X_\lambda'(1) X_\lambda(1) + X_\lambda'(0) X_\lambda(0) \geq 0.$$

Thus we have $\lambda \geq 0$ and furthermore

$$\lambda = 0 \text{ if and only if } X_\lambda' \equiv 0 \text{ and } \frac{\beta_1}{\alpha_1 + \beta_1} X_\lambda^2(0) + \frac{\beta_2}{\alpha_2 + \beta_2} X_\lambda^2(1) = 0,$$

or equivalently

$$X_\lambda \equiv C \text{ and } \Big(\frac{\beta_1}{\alpha_1+\beta_1} + \frac{\beta_2}{\alpha_2+\beta_2}\Big)C^2 = 0.$$

We can see from this expression that if $\beta_1+\beta_2>0$, then $X_\lambda \equiv 0$ which can not be a valid eigenfunction. In the end, we get that in the case of $\beta_1+\beta_2>0$, λ must be positive.

(2) We have already proved that $\lambda \geq 0$ and $\lambda = 0$ iff $\beta_1=\beta_2=0$. From $X''+\lambda X=0$, we know that for all constants A and B, the following representation is the solution,

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x,$$

$$X'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x.$$

We will get further information on the constants A and B from the boundary conditions. It will be studied in the following three cases:

(a) **Dirichlet boundary condition** $\alpha_1=\alpha_2=0$, then the boundary condition is reduced into $X(0)=X(1)=0$. Simple computations show that $A=0$ and $B\sin\sqrt{\lambda}=0$. As a consequence,

$$\lambda_n = (n\pi)^2, \quad X_n(x) = \sin n\pi x, \quad n = 1,2,\cdots.$$

Obviously, in this case, λ_n is monotone and increasing to ∞.

(b) **Neumann boundary condition** $\beta_1=\beta_2=0$, in this case the boundary condition is $X'(0)=X'(1)=0$, which implies that $B=0$, $A\sin\sqrt{\lambda}=0$. Thus we have

$$\lambda_n = (n\pi)^2, \quad X_n(x) = \cos n\pi x, \quad n = 0,1,2,\cdots.$$

(c) **Robin boundary condition** $\alpha_1\beta_2+\alpha_2\beta_1>0$. From the boundary condition we have

$$0 = \beta_1 A - \alpha_1 B\sqrt{\lambda},$$
$$0 = \beta_2(A\cos\sqrt{\lambda} + B\sin\sqrt{\lambda}) - \alpha_2\sqrt{\lambda}(A\sin\sqrt{\lambda} - B\cos\sqrt{\lambda}),$$

which implies that

$$\frac{\alpha_1\sqrt{\lambda}}{\beta_1} = \frac{A}{B},$$
$$0 = \beta_2\alpha_1\sqrt{\lambda}\frac{1}{\tan\sqrt{\lambda}} + \beta_2\beta_1 - \alpha_1\alpha_2\lambda + \beta_1\alpha_2\sqrt{\lambda}\frac{1}{\tan\sqrt{\lambda}}.$$

Let $\xi = \sqrt{\lambda}$, then we are left to solve the following equation,

$$\tan\xi = \frac{(\beta_2\alpha_1+\beta_1\alpha_2)\xi}{\alpha_1\alpha_2\xi^2 - \beta_1\beta_2}.$$

It can be proved that the sequence of all the possible solutions $\{\xi_n = \sqrt{\lambda_n}\}$ is monotone increasing and goes to ∞.

(3) Let λ, μ be two different eigenvalues. Multiplying the equation $X''_\lambda + \lambda X_\lambda = 0$ by X_μ and $X''_\mu + \mu X_\mu = 0$ by X_λ, integrating over $(0,1)$, we have

$$X_\mu X'_\lambda\Big|_0^1 - \int_0^1 X'_\mu X'_\lambda + \lambda \int_0^1 X_\lambda X_\mu = 0,$$
$$X_\lambda X'_\mu\Big|_0^1 - \int_0^1 X'_\mu X'_\lambda + \mu \int_0^1 X_\lambda X_\mu = 0.$$

The difference between this two equations shows that

$$(\lambda - \mu)\int_0^1 X_\lambda X_\mu = -X_\mu X'_\lambda\Big|_0^1 + X_\lambda X'_\mu\Big|_0^1.$$

Now the boundary conditions for X_λ and X_μ are

$$-\alpha_1 X'_\lambda(0) + \beta_1 X_\lambda(0) = 0, \qquad \alpha_2 X'_\lambda(1) + \beta_2 X_\lambda(1) = 0$$
$$-\alpha_1 X'_\mu(0) + \beta_1 X_\mu(0) = 0, \qquad \alpha_2 X'_\mu(1) + \beta_2 X_\mu(1) = 0$$

These algebraic systems have non zero solutions, thus the coefficient determinants are 0, i.e.,

$$\begin{vmatrix} X'_\lambda(0) & X_\lambda(0) \\ X'_\mu(0) & X_\mu(0) \end{vmatrix} = 0, \qquad \begin{vmatrix} X'_\lambda(1) & X_\lambda(1) \\ X'_\mu(1) & X_\mu(1) \end{vmatrix} = 0.$$

Thus we have

$$(\lambda - \mu)\int_0^1 X_\mu X_\lambda dx = 0.$$

Since $\lambda \neq \mu$, we know that X_λ and X_μ are orthogonal.

□

Remark 2.3. When $\beta_1 = \beta_2 = 0$, it is Neumann boundary condition. In this case, $\lambda = 0$ is an eigenvalue, and its eigenfunction is $X_0 = 1$.

Remark 2.4. $\{X_n(x)\}$ is a complete orthogonal basis of $L^2(0,1)$. After normalization, it is

$$X_n^*(x) = \frac{X_n(x)}{\|X_n(x)\|_{L^2}}.$$

Then $\forall f \in L^2(0,1)$, the fourier coefficients C_n^* are

$$C_n^* = \frac{\int_0^1 f(x) X_n(x) dx}{\|X_n(x)\|_{L^2}},$$

which is the inner product of $f(x)$ and $X_n^*(x)$.

2.3.3. *Solution formula by separation of variables.* Now we come back to the solution formula for (2.14) by using the method of separation of variables,

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos n\pi t + B_n \sin n\pi t) \sin n\pi x. \tag{2.16}$$

We only need to determine the coefficients A_n and B_n by using initial data. Take $t = 0$ in (2.16), we have

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin n\pi x \quad \text{and} \quad u_t(x,0) = \sum_{n=1}^{\infty} n\pi B_n \sin n\pi x.$$

For the initial data $g, h \in L^2(0,1)$, they have the following Fourier expansion by sine functions

$$g(x) = \sum_{n=1}^{\infty} g_n \sin n\pi x, \qquad g_n = 2\int_0^1 g(x) \sin n\pi x dx,$$
$$h(x) = \sum_{n=1}^{\infty} h_n \sin n\pi x, \qquad h_n = 2\int_0^1 h(x) \sin n\pi x dx,$$

then the coefficients can be determined by

$$A_n = g_n, \quad B_n = \frac{1}{n\pi} h_n.$$

Thus the solution expression is

$$u(x,t) = \sum_{n=1}^{\infty} (g_n \cos n\pi t + \frac{h_n}{n\pi} \sin n\pi t) \sin n\pi x. \tag{2.17}$$

Now we give a short summary on the three main steps in applying the method of separation of variables:

(1) let the solution have a factorized form $u(x,t) = X(x)T(t)$, and set up the eigenvalue problem,
(2) solve the eigenvalue problem, and solve the ODE for $T(t)$,
(3) take a summation of all the factorize-typed solutions, and fix the coefficients by using initial data.

After the above steps, we arrive at a solution formula, there are still a few questions left:

(1) How can one handle the inhomogeneous equation $u_{tt} - u_{xx} = f$?
(2) How can one deal with the inhomogeneous boundary conditions?
(3) Does the formula give us a classical solution? Under what condition is it a classical solution?

We will answer the remaining questions in the following.

Inhomogeneous equation We briefly explain how to deal with the inhomogeneous equations and left the details to the readers. Here we use $(0,l)$ instead of $(0,1)$.

$$\begin{aligned} u_{tt}-u_{xx}&=f(x,t), \qquad x\in(0,l), t>0,\\ u|_{x=0}=u|_{x=l}&=0, \\ u|_{t=0}=g(x), &\qquad u_t|_{t=0}=h(x). \end{aligned} \tag{2.18}$$

Firstly we know that the eigenfunctions are $\sin\dfrac{n\pi x}{l}$, $n=1,2,\cdots$. Then assume that

$$\begin{aligned} u(x,t)&=\sum_{n=1}^{\infty}T_n(t)\sin\frac{n\pi}{l}x,\\ f(x,t)&=\sum_{n=1}^{\infty}f_n(t)\sin\frac{n\pi}{l}x,\\ g(x)&=\sum_{n=1}^{\infty}g_n\sin\frac{n\pi}{l}x,\\ h(x)&=\sum_{n=1}^{\infty}h_n\sin\frac{n\pi}{l}x. \end{aligned}$$

Then solve the ODE for $T_n(t)$,

$$\begin{aligned} &T_n''(t)+(\frac{n\pi}{l})^2T_n(t)=f_n(t),\\ &T_n(0)=g_n, \quad T_n'(0)=h_n. \end{aligned}$$

One can get that the solution is

$$T_n(t)=g_n\cos\frac{n\pi}{l}t+\frac{l}{n\pi}h_n\sin\frac{n\pi}{l}t+\frac{l}{n\pi}\int_0^t f_n(\tau)\sin\frac{n\pi}{l}(t-\tau)d\tau.$$

Then by replacing $T_n(t)$ in the solution $u(x,t)$, we get the solution for inhomogeneous equation.

Inhomogeneous boundary conditions

The problem with inhomogeneous boundary condition is

$$\begin{aligned} u_{tt}-u_{xx}&=f, \qquad x\in(0,l), t>0,\\ u|_{x=0}&=u_0(t), \qquad u|_{x=l}=u_1(t),\\ u|_{t=0}&=g(x), \qquad u_t|_{t=0}=h(x). \end{aligned} \tag{2.19}$$

We will use homogenization technic. In other words, by introducing a new function $v(x,t)$ such that the homogeneous boundary conditions work for $v(x,t)$. More precisely, let

$$v(x,t):=u(x,t)-\frac{x}{l}u_1(t)-\frac{l-x}{l}u_0(t),$$

then $v(x,t)$ solves the following initial boundary value problem

$$\begin{aligned}
&v_{tt} - v_{xx} = f(x,t) - \frac{x}{l}u_1'' - \frac{l-x}{l}u_0'',\\
&v|_{x=0} = v|_{x=l} = 0,\\
&v|_{t=0} = g(x) - \frac{x}{l}u_1(0) - \frac{l-x}{l}u_0(0),\\
&v_t|_{t=0} = h(x) - \frac{x}{l}u_1'(0) - \frac{l-x}{l}u_0'(0).
\end{aligned}$$

By the method of dealing with inhomogeneous equations, we can get a formula for $v(x,t)$, which gives directly the expression of $u(x,t)$.

2.3.4. *Existence of solution for* (2.14). Under what conditions for the given data is the series in (2.17) exactly the classical solution of (2.14)?

If we can prove that u is at least twice differentiable in both x and t, then all the previous computations can go through, which means that it is indeed a classical solution. Therefore, we are left to prove that

$$\sum_{n=1}^{\infty} u_n, \quad \sum_{n=1}^{\infty} Du_n, \quad \sum_{n=1}^{\infty} D^2u_n$$

are uniformly convergent in $(0,1)\times(0,T)$.

In order to obtain the classical solution of (2.14), we require the following compatibility conditions

$$g(0) = g(1) = 0, \quad h(0) = h(1) = 0, \quad g''(0) = g''(1) = 0. \tag{2.20}$$

Theorem 2.6. *If $g \in C^3[0,1]$, $h \in C^2[0,1]$ and they satisfy the compatibility condition* (2.20)*, then $u(x,t)$ given by* (2.17) $= \sum_{n=1}^{\infty} u_n(x,t) \in C^2(\bar{Q})$ *is a solution of* (2.14).

Proof. Doing integration by parts on the coefficients of g and h by using compatibility conditions, we have

$$\begin{aligned}
\frac{h_n}{n\pi} = \frac{2}{n\pi}\int_0^1 h(x)\sin n\pi x dx = -\frac{2}{(n\pi)^3}\int_0^1 h''(x)\sin n\pi x dx \quad &=: \quad -\frac{2}{(n\pi)^3}a_n,\\
g_n = 2\int_0^1 g(x)\sin n\pi x dx = \frac{2}{(n\pi)^3}\int_0^1 g'''(x)\cos n\pi x dx \quad &=: \quad \frac{2}{(n\pi)^3}b_n.
\end{aligned}$$

Therefore

$$u(x,t) = \sum_{n=1}^{\infty}\Big(\frac{2}{(n\pi)^3}b_n\cos n\pi t - \frac{2}{(n\pi)^3}a_n\sin n\pi t\Big)\sin n\pi x.$$

Moreover the following estimates hold

$$\begin{aligned}
&|u_n| \le \frac{C}{n^3}, \quad |Du_n| \le \frac{C}{n^2},\\
&|D^2u_n| \le \frac{C}{n}(|a_n| + |b_n|) \le C(\frac{1}{n^2} + |a_n|^2 + |b_n|^2),
\end{aligned}$$

where the right hand side of the last inequality can be bounded by Bessel's inequality,

$$\sum_{n=1}^{\infty}|a_n|^2 \le 2\int_0^1 |h''|^2dx, \quad \sum_{n=1}^{\infty}|b_n|^2 \le 2\int_0^1 |g'''|^2dx.$$

□

2.3.5. *Uniqueness and Stability — Energy estimates.* Let $Q_\tau = (0,1)\times(0,\tau)$. We have the following energy estimates for initial boundary value problem of wave equation (2.14) in Q_τ. Then uniqueness and stability can be obtained from that.

Theorem 2.7. *Assume $u \in C^2(Q_\tau)\cap C^1(\bar{Q}_\tau)$ is a solution of (2.14), then*

$$\int_0^1 (u^2+u_t^2+u_x^2)dx \le M(\int_0^1 (h^2+g^2+g_x^2)dx + \int_{Q_\tau} f^2dxdt).$$

Proof. Multiplying the wave equation by u_t and integrating it over Q_τ gives

$$\int_{Q_\tau}\frac{\partial}{\partial t}(u_t^2+u_x^2) \le \int_{Q_\tau} f^2 + \int_{Q_\tau} u_t^2.$$

Notice that $\int_{Q_\tau} = \int_0^\tau\int_0^1$, we have

$$\int_0^1 (u_t^2+u_x^2)|_{t=\tau} \le \int_0^1 (h^2+g_x^2) + \int_{Q_\tau} f^2 + \int_{Q_\tau} u_t^2.$$

By Gronwall's inequality,

$$\int_0^1 (u_t^2+u_x^2)|_{t=\tau} \le M\Big(\int_0^1 (h^2+g_x^2) + \int_{Q_\tau} f^2\Big).$$

Similar to the discussion in Cauchy problem, we have the L^2 estimates.

□

2.3.6. *Resonance.* In the solution formula, $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$, the n-th wave is a wave that remains in a constant position in the following sense

$$\begin{aligned}
u_n(x,t) &= (g_n\cos n\pi t + \frac{h_n}{n\pi}\sin n\pi t)\sin n\pi x\\
&= (g_n^2+(h_n/n\pi)^2)^{\frac12}\sin n\pi x\Big(\frac{g_n}{(g_n^2+(h_n/n\pi)^2)^{\frac12}}\cos n\pi t\\
&\qquad + \frac{h_n/n\pi}{(g_n^2+(h_n/n\pi)^2)^{\frac12}}\sin n\pi t\Big)\\
&= (g_n^2+(h_n/n\pi)^2)^{\frac12}\sin n\pi x\cdot\sin(n\pi t+\alpha_n)\\
&= N_n\cdot\sin n\pi x\cdot\sin(n\pi t+\alpha_n),
\end{aligned}$$

where $\tan\alpha_n = \dfrac{n\pi g_n}{h_n}$, $N_n = (g_n^2+(h_n/n\pi)^2)^{\frac12}$. Usually $N_1 \gg N_n$, $\forall n\neq 1$.

Consider the following initial boundary value problem

$$\begin{aligned} u_{tt} - u_{xx} &= A(x)\sin\omega t, \quad x \in (0,1), t > 0,\\ u|_{x=0,1} &= 0,\\ u|_{t=0} = u_t|_{t=0} &= 0. \end{aligned}$$

Compatibility condition $A(0) = A(1) = 0$ is needed for the existence of classical solution. We assume $A \in C^1$. The solution formula from separation of variables is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \frac{1}{n\pi}\int_0^t f_n(\tau)\sin n\pi(t-\tau)d\tau \cdot \sin n\pi x\\ &= \sum_{n=1}^{\infty} \frac{a_n}{n\pi}\sin n\pi x \int_0^t \sin\omega\tau \cdot \sin n\pi(t-\tau)d\tau, \end{aligned}$$

where

$$a_n = 2\int_0^1 A(x)\sin n\pi x dx.$$

If we calculate further, we will see

$$\begin{aligned} & \int_0^t \sin\omega\tau \cdot \sin n\pi(t-\tau)d\tau\\ = \;& \int_0^t -\frac{1}{2}(\cos(n\pi t + (\omega - n\pi)\tau) - \cos(-(\omega+n\pi)\tau + n\pi t))d\tau\\ = \;& \frac{1}{2}\int_0^t \cos((\omega+n\pi)\tau - n\pi t)d\tau - \frac{1}{2}\int_0^t \cos((\omega - n\pi)\tau + n\pi t)d\tau\\ \overset{\omega\neq n\pi}{=} \;& \frac{1}{2(\omega+n\pi)}\sin((\omega+n\pi)\tau - n\pi t)\Big|_0^t - \frac{1}{2(\omega-n\pi)}\sin((\omega-n\pi)\tau + n\pi t)\Big|_0^t\\ \overset{\omega\neq n\pi}{=} \;& \frac{1}{2(\omega+n\pi)}(\sin\omega t + \sin n\pi t) - \frac{1}{2(\omega-n\pi)}(\sin\omega t - \sin n\pi t). \end{aligned}$$

If $\omega = k\pi$ for some k, then

$$\begin{aligned} u_k(x,t) &= \frac{a_k}{k\pi}\Big(\frac{\sin k\pi t}{k\pi + k\pi} - \frac{1}{2}\int_0^t \cos k\pi t d\tau\Big)\sin k\pi x\\ &= \Big(\frac{a_k}{2(k\pi)^2}\sin k\pi t - \frac{a_k}{2k\pi}t\cdot\cos k\pi t\Big)\sin k\pi x. \end{aligned}$$

Thus in the case of $\omega = k\pi$, we have

$$\begin{aligned} u(x,t) &= \sum_{n\neq k}\frac{a_n}{n\pi}\Big(\frac{1}{2(\omega+n\pi)}(\sin\omega t + \sin n\pi t) - \frac{1}{2(\omega-n\pi)}(\sin\omega t - \sin n\pi t)\Big)\sin n\pi x\\ &\quad + \Big(\frac{a_k}{2(k\pi)^2}\sin k\pi t - \frac{a_k}{2k\pi}t\cdot\cos k\pi t\Big)\sin n\pi x \end{aligned}$$

Therefore there exists a sequence of time $t_m \to \infty$ such that the $u_k(x,t)$ is unbounded for big m.

2.4. **Appendix-On Fourier Series.** $\forall f \in L^1(-l,l)$, it can be written into a series by using trigonometric functions

$$f(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l}),$$

where

$$A_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx, \quad n = 0, 1, 2, \cdots,$$
$$B_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx, \quad n = 1, 2, \cdots$$

are called Fourier coefficients of f.

If $f(x)$ is an even function, then $B_n = 0$, and

$$f(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}, \quad A_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx.$$

If $f(x)$ is an odd function, then $A_n = 0$, and

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}, \quad B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx.$$

For any fixed $N \geq 1$, $(S_N f)(x) = \frac{A_0}{2} + \sum_{n=1}^{N} (A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l})$ is called trigonometric polynomial.

$\sin \frac{n\pi x}{l}$, $\cos \frac{m\pi x}{l}$, $n, m = 1, 2, \cdots$ are orthogonal in the following sense

$$\frac{1}{l} \int_{-l}^{l} \cos \frac{m\pi x}{l} \cos \frac{n\pi x}{l} dx = \delta_{mn},$$
$$\frac{1}{l} \int_{-l}^{l} \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l} dx = \delta_{mn},$$
$$\frac{1}{l} \int_{-l}^{l} \sin \frac{m\pi x}{l} \cos \frac{n\pi x}{l} dx = 0.$$

Moreover, $\{1, \sqrt{2} \cos \frac{n\pi x}{l}, \sqrt{2} \sin \frac{n\pi x}{l}\}_{n=1}^{\infty}$ is an orthonormal basis in $L^2(-l,l)$, where the inner product in $L^2(-l,l)$ is defined by $\frac{1}{2l} \int_{-l}^{l} f(x) \bar{g}(x) dx$, $\forall f, g \in L^2$.

Here we list several useful facts for trigonometric series that we are going to use in this course.

Theorem 2.8. *(Convergence in L^2 norm)*

$$\lim_{N \to \infty} \|f(x) - (S_N f)(x)\|_{L^2} = 0, \quad \textit{for } f \in L^2(-l,l).$$

Theorem 2.9. *(Bessel's inequality) For $f \in L^2(-l,l)$, it holds*

$$\frac{A_0^2}{2} + \sum_{n=1}^{\infty}(A_n^2 + B_n^2) \le \frac{1}{l}\int_{-l}^{l} f^2 dx.$$

Theorem 2.10. *(Parseval's equality) For $f \in L^2(-l,l)$, it holds*

$$\frac{A_0^2}{2} + \sum_{n=1}^{\infty}(A_n^2 + B_n^2) = \frac{1}{l}\int_{-l}^{l} f^2 dx.$$

2.5. ***Generalized solution.** It happens quite often that in some of the physical models one cannot find the solution in the classical sense. For example, in the model for the vibrating string, the initial data can be piecewise differentiable. A typical example is $g(x) = x$ for $x \in [0,1/2]$ and $g(x) = 1-x$ for $x \in [1/2,1]$. Now g is no longer a differentiable function, but the problem should still have a "reasonable" solution. One must search for the solution in a broader class.

In this subsection, we introduce the basic knowledge of weak solution for the following initial boundary value problem

$$\begin{aligned} u_{tt} - u_{xx} &= f, \qquad Q_T = (0,1)\times(0,T) \\ u|_{x=0} = u|_{x=1} &= 0, \\ u|_{t=0} &= g, \qquad u_t|_{t=0} = h. \end{aligned}$$

Here we give a motivation in defining the solution in a more generalized sense.

The motivation is the following. We multiply the equation by φ and integrate on the domain Q_T,

$$\int_{Q_T}(u_{tt} - u_{xx})\varphi = \int_{Q_T} f\varphi.$$

Integration by parts shows that

$$\int_{Q_T} u(\varphi_{tt} - \varphi_{xx}) + \int_0^1 u_t\varphi\Big|_0^T dx - \int_0^1 u\varphi_t\Big|_0^T dx + \int_0^T u_x\varphi\Big|_0^1 dt - \int_0^T u\varphi_x\Big|_0^1 dt = \int_{Q_T} f\varphi.$$

If we choose the test function φ to be the one that satisfies the boundary conditions

$$\varphi|_{t=T} = \varphi_t|_{t=T} = 0, \quad \varphi|_{x=0,1} = 0,$$

we will have

$$\int_{Q_T} u(\varphi_{tt} - \varphi_{xx}) + \int_0^1 h\varphi(x,0) - \int_0^1 g\varphi_t(x,0) = \int_{Q_T} f\varphi.$$

Notice that in the above equation, we don't require any smoothness of u, which means we can define the solution by using the above equation, which is of the integral form.

Now we will fix our test function in the set

$$\mathcal{D} = \{\varphi \in C^2(\bar{Q}_T) | \varphi(x,T) = \varphi_t(x,T) = 0, \varphi|_{x=0,1} = 0\}.$$

Definition 1. If $u \in C(\bar{Q})$ and $\forall \varphi \in \mathcal{D}$, there holds

$$\int_{Q_T} u(\varphi_{tt} - \varphi_{xx}) - \int_0^1 h\varphi(x,0) + \int_0^1 g\varphi_t(x,0) = \int_{Q_T} f\varphi, \tag{2.21}$$

then we call u is a weak solution of (2.18).

Remark 2.5. Due to the motivations, we know that classical solutions must be weak solutions.

Theorem 2.11. *(Uniqueness of weak solution) If u_1 and u_2 are two weak solutions of* (2.18)*, then $u_1 = u_2$ in Q_T.*

Proof. By definition, we know that $\forall \varphi \in \mathcal{D}$,

$$\int_{Q_T} (u_1 - u_2)(\varphi_{tt} - \varphi_{xx}) = 0.$$

"We want to show that$\forall \psi \in C_0^\infty(Q_T)$

$$\begin{aligned} &\varphi_{tt} - \varphi_{xx} = \psi, \\ &\varphi|_{x=0,1} = 0, \varphi|_{t=T} = 0, \varphi_t|_{t=T} = 0 \end{aligned}$$

has a solution. This is a backward wave equation, by changing variable $\tau = T - t$, let $\bar{\varphi}(x,\tau) = \varphi(x,t)$, the problem will change into

$$\begin{aligned} &\bar{\varphi}_{tt} - \bar{\varphi}_{xx} = \psi(x, T-\tau), \\ &\bar{\varphi}|_{x=0,1} = 0, \bar{\varphi}|_{t=T} = 0, \bar{\varphi}_t|_{t=T} = 0. \end{aligned}$$

Since $\psi \in C_0^\infty(Q_T)$ satisfies the compatibility conditions, from existence theory, we know that this problem has a solution $\varphi(x,t) \in C^2(\bar{Q}_T)$. Since ψ is arbitrary, we know that $u_1 = u_2$. □

Theorem 2.12. *(***Stability of weak solution) If $f \equiv 0$, then*

$$\int_{Q_T} (u_1 - u_2)^2 \le C\Big(\int_0^1 |g_1 - g_2|^2 + \int_0^1 |h_1 - h_2|^2\Big).$$

Proof. $\forall \varphi \in C_0^\infty(Q_T)$, by the definition of weak solution, we have

$$\int_{Q_T} (u_1 - u_2)(\varphi_{tt} - \varphi_{xx}) - \int_0^1 (h_1 - h_2)\varphi(x,0) + \int_0^1 (g_1 - g_2)\varphi_t(x,0) = 0. \tag{2.22}$$

Now we consider problem

$$\begin{aligned} &\varphi_{tt} - \varphi_{xx} = u_1^\varepsilon - u_2^\varepsilon, \\ &\varphi|_{x=0,1} = 0, \varphi|_{t=T} = 0, \varphi_t|_{t=T} = 0, \end{aligned}$$

where $u_1^\varepsilon, u_2^\varepsilon \in C_0^\infty(Q_T)$ and $u_i^\varepsilon \to u_i$ in $L^2(Q_T)$, $i = 1, 2$. By changing time variable $\tau = T - t$, we have

$$\begin{aligned} &\bar{\varphi}_{tt} - \bar{\varphi}_{xx} = (u_1^\varepsilon - u_2^\varepsilon)(x, T-\tau), \\ &\bar{\varphi}|_{x=0,1} = 0, \bar{\varphi}|_{t=T} = 0, \bar{\varphi}_t|_{t=T} = 0. \end{aligned}$$

Then by the existence of classical solution, we know that the above problem has a solution $\bar{\varphi}(x,\tau) \in C^2(\bar{Q}_T)$ and the energy estimates hold, i.e.,

$$\int_0^1 (|\bar{\varphi}|^2 + |\bar{\varphi}_\tau|^2)\Big|_{\tau=T} \le C\|u_1^\varepsilon - u_2^\varepsilon\|^2_{L^2(Q_T)} \le C\|u_1 - u_2\|^2_{L^2(Q_T)}.$$

Therefore putting them together into (2.22), we have

$$\int_{Q_T} (u_1 - u_2)(u_1^\varepsilon - u_2^\varepsilon) \le \|h_1 - h_2\|_{L^2} \cdot \|\varphi\|_{L^2} + \|g_1 - g_2\|_{L^2} \cdot \|\varphi_t\|_{L^2}.$$

Taking $\varepsilon \to 0$, we have

$$\int_{Q_T} (u_1 - u_2)^2 \le C(\|h_1 - h_2\|_{L^2} + \|g_1 - g_2\|_{L^2}) \cdot \|u_1 - u_2\|_{L^2}.$$

This completes the proof. □

Theorem 2.13. *(Existence of weak solution) Let $g \in C[0,1]$, $g(0) = g(1) = 0$, g' and h are piecewise continuous in $[0,1]$, then*

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

is a weak solution of (2.14).

Proof. Fix an integer $N > 0$, let

$$G_N(x) = \sum_{n=1}^{N} g_n \sin n\pi x, \quad H_N(x) = \sum_{n=1}^{N} h_n \sin n\pi x.$$

Now we consider the approximation problem

$$\begin{aligned} \partial_{tt} u_N - \partial_{xx} u_N &= 0, \quad \text{in } (0,1)\times(0,T], \\ u_N|_{x=0,1} &= 0, \\ u_N|_{t=0} &= G_N, \quad \partial_t u_N|_{t=0} = H_N. \end{aligned} \tag{2.23}$$

Then by separation of variables,

$$u_N = \sum_{n=1}^{N} (g_n \cos n\pi t + \frac{h_n}{n\pi} \sin n\pi t) \sin n\pi x$$

is a classical solution of (2.23).

For any test function $\varphi \in \mathcal{D}$, we have

$$\int_{Q_T} u_N(\varphi_{tt} - \varphi_{xx}) + \int_0^1 G_N(x)\varphi_t(x,0) - \int_0^1 H_N(x)\varphi(x,0) = 0. \tag{2.24}$$

If $g(x)$, $h(x) \in L^2(0,1)$, we know that

$$G_N(x) \to g(x), \quad H_N(x) \to h(x), \quad \text{in } L^2.$$

Moreover,

$$\begin{aligned}|u_N| &\leq \sum_{n=1}^{N}\Big(|g_n|+\Big|\frac{h_n}{n\pi}\Big|\Big)\leq \sum_{n=1}^{N}\frac{C}{n}(|(g')_n|+|h_n|)\\ &\leq C\sum_{n=1}^{N}\Big(\frac{1}{n^2}+|(g')_n|^2+|h_n|^2\Big).\end{aligned}$$

So we have that

$$u_N \to u=\sum_{n=1}^{\infty}u_n, \qquad \text{uniformly in } (0,1).$$

Now we take limit $N\to\infty$ in (2.25), it follows

$$\int_{Q_T} u(\varphi_{tt}-\varphi_{xx})+\int_0^1 g(x)\varphi_t(x,0)-\int_0^1 h(x)\varphi(x,0)=0.$$

□

Remark 2.6. In the above proof, the condition that g' and $h\in L^2(0,1)$ is sufficient to guarantee the existence of weak solution.

2.6. **Problems.**

(1) Verify that $u(x,t)=\dfrac{F(x-at)+G(x+at)}{h-x}$ is a solution of

$$\Big(1-\frac{x}{h}\Big)^2\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial}{\partial x}\Big[\Big(1-\frac{x}{h}\Big)^2\frac{\partial u}{\partial x}\Big]$$

where $h>0$, $a>0$ are constants, F,G are any function in C^2.

(2) (a) Show the general solution of the PDE $u_{xy}=0$ is $u(x,t)=F(x)+G(y)$ for arbitrary function F,G.
 (b) Using the change of variables $\xi=x+t$, $\eta=x-t$, show $u_{tt}-u_{xx}=0$ if and only if $u_{\xi\eta}=0$.
 (c) Use the above two facts to derive d'Alembert's formula.

(3) Give energy estimates for half-line problem and the Cauchy problem in Multi-D case.

(4) (Equal partition of energy) Suppose that $u\in C^2(\mathbb{R}\times[0,\infty))$ is a solution of the following Cauchy problem

$$\begin{aligned} u_{tt}-u_{xx}=0, &\qquad (x,t)\in\mathbb{R}\times(0,\infty),\\ u|_{t=0}=g,\quad u_t|_{t=0}=h, &\qquad x\in\mathbb{R},\end{aligned}$$

where g and h have compact support. Let kinetic energy be $k(t)=\dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty}u_t^2(x,t)dx$, potential energy be $p(t)=\dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty}u_x^2(x,t)dx$. Try to prove
 (a) $k(t)+p(t)$ is a constant independent of t.
 (b) $k(t)=p(t)$ for large enough t.

(5)

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \qquad x \in (0, +\infty), t \in (0, +\infty), \\ u|_{x=0} &= \cos \omega t, \\ u|_{t=0} &= Ae^{-x^2}, \qquad u_t|_{t=0} = 0. \end{aligned}$$

Find the condition for A and ω such that solution $u \in C^2(\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+)$, and give this solution formula.

(6) If u is a classical solution of

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \qquad x \in (0,1), t \in (0, +\infty), \\ u|_{x=0} = u|_{x=1} &= 0, \\ u|_{t=0} &= 0, \qquad u_t|_{t=0} = x^2(1-x), \end{aligned}$$

try to compute the limit

$$\lim_{t \to +\infty} \int_0^1 (u_t^2 + u_x^2) dx.$$

(7) Solve the eigenvalue problem

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, \quad x \in (0, l), \\ X(0) = X'(l) &= 0. \end{aligned}$$

(8)

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, \quad x \in (0,1), \\ X'(0) + X(0) &= 0, \quad X(1) = 0. \end{aligned}$$

(a) Find an eigenfunction with eigenvalue zero. Call it $X_0(x)$.
(b) Find an equation for the positive eigenvalues $\lambda = \beta^2$.
(c) Show graphically from part (8b) that there are an infinite number of positive eigenvalues.
(d) Is there a negative eigenvalue?

(9) Apply separation of variables to get formal solution of

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \qquad (x,t) \in (0,1) \times (0, \infty), \\ u_x|_{x=0} = A \sin \omega t, u|_{x=1} &= 0, \quad t \geq 0, \\ u|_{t=0} = 1, \quad u_t|_{t=0} &= 0, \qquad x \in [0,1]. \end{aligned}$$

(10)

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \qquad x \in (0,1), t \in (0, +\infty), \\ u|_{x=0} = u|_{x=1} &= 0, \\ u|_{t=0} = \alpha x^4 + \beta x^3 + \sin x, &\qquad u_t|_{t=0} = \gamma \cos x. \end{aligned}$$

Solve the problem and give the conditions on α, β and γ such that the solution you gave is a classical one.

(11) Find the solution of initial boundary values for heat equation by separation of variables.

$$\begin{aligned} u_t - u_{xx} &= \sin x\pi, \qquad x \in (0,1), t \in (0,+\infty), \\ u|_{x=0} = u|_{x=1} &= 0, \\ u|_{t=0} &= 0. \end{aligned}$$

(12) **Discussions** One can get solution of (2.14) by d'Alembert' formula and Fourier series. Are they the same?

3. Heat equation

We will first give an introcduction for Fourier tansform and distribution, together with a list of their properties that will be used within this course. The solution for Cauchy problem of heat equation will be given by using heat kernel. Furthermore, the solution formula for half space problem will be derived by using Green's function. As further understanding of the Fourier series and energy methods, the initial boundary value problem for heat equation will be also be briefly studied. We will also introduce a new method, the maximum principle, for parabolic and elliptic type of equations. It provides the first stage a priori estimate and also can be used to study the uniqueness and stability of the solutions. In the end, we will study the long time behavior of the solution of heat equation, with Dirichlet and Neumann boundary conditions separately.

3.1. **A short introduction on Fourier transform and distribution.**

3.1.1. *Fourier transform.* Let's first recall the Fourier series. $\forall f \in L^1(-l,l)$, its Fourier series is defined by

$$f(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}),$$

where

$$A_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx, \quad n = 0,1,2,\cdots,$$
$$B_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx, \quad n = 1,2,\cdots.$$

By Euler's formula, we can change the items in the summation into

$$A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l} = A_n' e^{i\frac{n\pi x}{l}} + B_n' e^{-i\frac{n\pi x}{l}}.$$

Thus the Fourier series can be rewritten into

$$f(x) \sim \sum_{-\infty}^{\infty} a_n e^{i\frac{n\pi x}{l}}, \qquad a_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-i\frac{n\pi x}{l}}dx,$$

i.e.,

$$f(x) \sim \frac{1}{2l}\sum_{-\infty}^{\infty}\int_{-l}^{l} f(y)e^{-i\frac{n\pi y}{l}}dye^{i\frac{n\pi x}{l}}.$$

Now let $k_n = \frac{n\pi}{l}$, the formula is reduced into

$$f(x) \sim \frac{1}{2\pi}\sum_{-\infty}^{\infty}\int_{-l}^{l} f(y)e^{-ik_n y}dye^{ik_n x}\frac{\pi}{l}.$$

Formally letting $l \to \infty$, one could expect that

$$f(x) \sim \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)e^{-iky}dye^{ikx}dk.$$

These formal computations give the motivation in the definition of Fourier transform on $\mathbb{R}$.

Definition 2. $\forall f \in L^1(\mathbb{R})$, its Fourier transform is defined by

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx.$$

A result which can be obtained directly from the definition is that $\hat{f}(k) \in L^\infty(\mathbb{R})$, i.e.,

$$|\hat{f}(k)| = \frac{1}{\sqrt{2\pi}}\left|\int_{-\infty}^{\infty} f(x)e^{-ikx}dx\right| \le \frac{1}{\sqrt{2\pi}}\|f\|_{L^1}.$$

Therefore the definition tells that Fourier transform is a continuous linear mapping from L^1 to L^∞. Moreover, the following result holds

Theorem 3.1. *If $f \in L^1(\mathbb{R})$, then $\hat{f}(k)$ is uniformly continuous in $\mathbb{R}$.*

Proof. We prove it directly by using the definition of uniformly continuous. $\forall \varepsilon > 0$, $f \in L^1(\mathbb{R})$ implies that $\exists A > 0$, such that

$$\frac{1}{\sqrt{2\pi}}\int_{|x|>A} 2|f|dx \le \frac{\varepsilon}{2}.$$

$\forall 0 < h < \dfrac{\sqrt{2\pi}\varepsilon}{4A\|f\|_{L^1}}$, we have

$$\begin{aligned}
|\hat{f}(k+h) - \hat{f}(k)| &= \frac{1}{\sqrt{2\pi}}\left|\int_{-\infty}^{\infty} f(x)e^{-ixk}[e^{-ixh} - 1]dx\right| \\
&\le \frac{1}{\sqrt{2\pi}}\int_{|x|>A} 2|f|dx + \frac{1}{\sqrt{2\pi}}\int_{-A}^{A} |x|\cdot|h|\cdot|f|dx \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

□

Remark 3.1. Similarly, one can define Fourier transform in multi-D case, $\forall f \in L^2(\mathbb{R}^n)$,

$$\hat{f}(k) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{R^n} f(x)e^{-ik\cdot x}dx,$$

where $k \cdot x = \sum_{i=1}^n k_i x_i$. It is also a continuous linear mapping from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.

We list here a series of properties of Fourier transform without proof.

(1) $(\partial_{x_j} f)^\wedge(k) = ik_j\hat{f}(k)$, and $(\partial^\alpha f)^\wedge(k) = i^{|\alpha|}k^\alpha\hat{f}(k)$,
(2) $(x_j f)^\wedge(k) = i\partial_{k_j}\hat{f}(k)$, and $(x^\alpha f)^\wedge(k) = i^{|\alpha|}\partial^\alpha\hat{f}(k)$,
(3) $f(x-a)^\wedge(k) = e^{-ia\cdot k}\hat{f}(k)$,
(4) $(f(\lambda x))^\wedge(k) = \dfrac{1}{|\lambda|^n}\hat{f}(\dfrac{k}{\lambda})$, $\forall \lambda \neq 0$,

(5) $(f*g)^\wedge(k) = (2\pi)^{\frac{n}{2}}\hat{f}(k)\hat{g}(k)$.

Example 1. Fourier transform of Gaussian e^{-x^2} in 1-d is $\dfrac{1}{\sqrt{2}}e^{-\frac{k^2}{4}}$. More general case in multi-dimension is

$$(e^{-A|x|^2})^\wedge(k) = \prod_1^n (e^{-Ax_i^2})^\wedge(k_i) = \frac{1}{(2A)^{\frac{n}{2}}}e^{-\frac{|k|^2}{4A}}, \quad \forall A > 0.$$

The **inverse Fourier transform** can be formally given by

$$\check{f}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(k)e^{ik\cdot x}dk.$$

The following theorem assures that the inverse Fourier transform is also well-defined.

Theorem 3.2. *If $f \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$, then*

$$\lim_{N\to\infty}\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-N}^{N}\hat{f}(k)e^{ikx}dk = f(x).$$

Proof. *(For those who are interested) We know that $\hat{f}(k)$ is uniformly bounded and continuous in $k \in \mathbb{R}$, by the definition of Fourier transform, we have

$$\begin{aligned}
&\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-N}^{N}\hat{f}(k)e^{ikx}dk \\
= &\frac{1}{2\pi}\int_{-N}^{N}\int_{-\infty}^{\infty} f(y)e^{-iky}dye^{ikx}dk \\
= &\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\int_{-N}^{N} e^{ik(x-y)}dk\Big)f(y)dy,
\end{aligned}$$

where

$$\int_{-N}^{N} e^{ik(x-y)}dk = 2\frac{\sin N(x-y)}{x-y}.$$

This is similar to the Dirichlet kernel, therefore one can expect that the whole integral will converge to $f(x)$ as $N \to \infty$. Next we will prove it in detail.

Changing variable $z = y - x$ gives

$$\begin{aligned}
&\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-N}^{N}\hat{f}(k)e^{ikx}dk \\
= &\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin N(x-y)}{x-y}f(y)dy = \frac{1}{\pi}\int_{-\infty}^{\infty} f(z+x)\frac{\sin Nz}{z}dz.
\end{aligned}$$

Now we separate the integral on $\mathbb{R}$ into two parts $I_1 = \displaystyle\int_{|z|\le M}$ and $I_2 = \displaystyle\int_{|z|\ge M}$, where M is to be determined later. In the following, we will estimate I_1 by using Riemann' lemma, while the estimate for I_2 by $1/M$.

$\forall\varepsilon > 0$, choose $M = \dfrac{2\|f\|_{L^1}}{\pi\varepsilon}$, we have

$$I_2 = \frac{1}{\pi}\int_{|z|\geq M} f(z+x)\frac{\sin Nz}{z}dz \leq \frac{1}{\pi M}\|f\|_{L^1} = \frac{\varepsilon}{2}.$$

The way to estimate I_1 is by using

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}dx = \pi,$$

from which we know that $\exists N \geq 0$ s.t.

$$\Big|\frac{f(x)}{\pi}\int_{-MN}^{MN}\frac{\sin z}{z}dz - f(x)\Big| \leq \frac{\varepsilon}{4}.$$

Now I_1 can be estimated in the following way,

$$\begin{aligned}
I_1 &= \frac{1}{\pi}\int_{|z|\leq M} f(z+x)\frac{\sin Nz}{z}dz \\
&= \frac{1}{\pi}\int_{|z|\leq M}\frac{f(z+x)-f(x)}{z}\sin Nz dz + \frac{f(x)}{\pi}\int_{|z|\leq M}\frac{\sin Nz}{z}dz \\
&= \frac{1}{\pi}\int_{|z|\leq M}\int_0^1 f'(x+\tau z)d\tau \sin Nz dz + \frac{f(x)}{\pi}\int_{|z|\leq M}\frac{\sin Nz}{z}dz \\
&\leq \frac{\|f'\|_{L^\infty}}{\pi}\int_{|z|\leq M}\sin Nz dz + \frac{f(x)}{\pi}\int_{|z|\leq M}\frac{\sin Nz}{z}dz.
\end{aligned}$$

By Riemann's lemma, we know $\exists N_1 > 0$ s.t. when $N \geq N_1$, we have

$$\frac{\|f'\|_{L^\infty}}{\pi}\int_{|z|\leq M}\sin Nz dz \leq \frac{\varepsilon}{4}.$$

□

Fourier transform for L^2-functions

Theorem 3.3. *If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\hat{f} \in L^2(\mathbb{R})$ and*

$$\|\hat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}.$$

Furthermore, $f \to \hat{f}$ has a unique extension to a continuous, linear map from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, which is isometry.

Proof. Due to the fact that C_0^∞ is dense in L^p, for $p \geq 1$. We only need to prove the equation for all $f \in L^\infty$. $\forall\varepsilon > 0$, we consider the following weighted integral

$$\int_{\mathbb{R}}|\hat{f}(k)|^2 e^{-\varepsilon|k|^2}dk.$$

By the definition of Fourier transform, we have

$$\int_{\mathbb{R}}|\hat{f}(k)|^2 e^{-\varepsilon|k|^2}dk = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\overline{f(x)}f(y)e^{ik(x-y)}e^{-\varepsilon|k|^2}dxdydk.$$

Due to the fact that $\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\varepsilon k^2} e^{ik(x-y)}dk = (e^{-\varepsilon k^2})^{\vee}(x-y)$, with the help of Fubini theorem, the above integral is exactly

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\sqrt{2\varepsilon}} e^{-\frac{(x-y)^2}{4\varepsilon}}\overline{f(x)}f(y)dxdy.$$

Since

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-(\frac{x}{2\sqrt{\varepsilon}})^2} d(\frac{x}{2\sqrt{\varepsilon}}) = 1,$$

It is easy to check that for $f \in C_0^\infty$,

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{1}{\sqrt{2\varepsilon}} e^{-\frac{(x-y)^2}{4\varepsilon}} f(y)dy \to f(x), \quad \text{for all } x \in \mathbb{R}.$$

Thus

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} |\hat{f}(k)|^2 e^{-\varepsilon|k|^2}dk = \int_{\mathbb{R}} |f|^2 dx.$$

Then monotone convergence theorem implies that $\hat{f} \in L^2$ and

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2}.$$

If $f \in L^2$ but not in C_0^∞, since C_0^∞ is dense in L^2, there exists $\{f_j\} \subset C_0^\infty$ such that

$$\|f_j - f\|_{L^2} \to 0.$$

On the other hand, since Fourier transform is linear, we have

$$\|\hat{f}_j - \hat{f}_m\|_{L^2} = \|f_j - f_m\|_{L^2} \to 0, \text{ as } j, m \to \infty.$$

Hence, $\{\hat{f}_j\}$ is a Cauchy sequence in L^2. Since L^2 is a complete space, we have $\exists g \in L^2$ such that $\hat{f}_j \to g$ strongly in L^2.

Now we define $\hat{f} = g$, thus we have

$$\|\hat{f}\|_{L^2} = \lim_{j\to\infty}\|\hat{f}_j\|_{L^2} = \lim_{j\to\infty}\|f_j\|_{L^2} = \|f\|_{L^2}.$$

Continuity and linearity are left to readers. □

Remark 3.2. Fourier transform can be extended for L^p functions in a similar way by using the following inequality

$$\|\hat{f}\|_{L^q} \le C(p,q)\|f\|_{L^p}, \ \frac{1}{p} + \frac{1}{q} = 1.$$

3.1.2. *Distribution and weak derivative.* Let Ω be an open subset of $\mathbb{R}^n$.

Distributions

Definition 3. Test function space $\mathcal{D}(\Omega)$ consists of all the functions in $C_0^\infty(\Omega)$ supplemented by the following convergence: $\phi_m \to \phi \in C_0^\infty(\Omega)$ iff

(1) $\exists$ a compact set $K \subset \Omega$ such that $\text{supp}\phi_m \subset K$, $\forall m$,

(2) for all multi-index $\forall\alpha$,

$$\sup_K |\partial^\alpha \phi_m - \partial^\alpha \phi| \to 0, \text{ as } m \to \infty.$$

Remark 3.3. $\mathcal{D}(\Omega)$ is a linear space.

Definition 4. Distribution, denoted by $\mathcal{D}'(\Omega)$, is the dual space of $\mathcal{D}(\Omega)$, i.e., is the linear space that includes all the continuous linear functionals on $\mathcal{D}(\Omega)$. $T : \mathcal{D}(\Omega) \to \mathbb{C}$ is called continuous linear iff

(1) $\langle T, \alpha\phi_1 + \beta\phi_2\rangle = \alpha\langle T, \phi_1\rangle + \beta\langle T, \phi_2\rangle$,
(2) If $\phi_m \to \phi$ in $\mathcal{D}(\Omega)$, then $\langle T, \phi_m\rangle \to \langle T, \phi\rangle$.

Remark 3.4. It is usually nonsense to multiply two distributions, since it is not well defined.

Remark 3.5. A distribution multiplied by a smooth function can be defined by the following. For $T \in \mathcal{D}'$ and $f \in C^\infty$, their product is defined by

$$\langle Tf, \phi\rangle = \langle T, f\phi\rangle, \qquad \forall\phi \in \mathcal{D}.$$

Remark 3.6. The support of a distribution and the convolution of two distributions can be also defined with the help of test functions, since we will not use these in our lecture, we omit the detail here.

Example 2. $L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$.

$\forall f \in L^1_{loc}(\Omega)$, $T_f \in \mathcal{D}'(\Omega)$ is defined by

$$\langle T_f, \phi\rangle = \int_\Omega f(x)\phi(x)dx, \qquad \forall\phi \in \mathcal{D}(\Omega).$$

Remark 3.7. Similarly, $L^p_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$. One can use the Hölder's inequality to obtain that $L^p_{loc}(\Omega) \subset L^q_{loc}(\Omega)$ for $1 < q < p < \infty$.

Theorem 3.4. *L^1_{loc} functions are uniquely determined by distributions. More precisely, for two functions $f, g \in L^1_{loc}(\Omega)$, if*

$$\int_\Omega f\phi dx = \int_\Omega g\phi dx, \qquad \forall\phi \in \mathcal{D}(\Omega),$$

then $f = g$ a.e. in Ω.

The proof is left to the readers.

Example 3. Probability density functions on $\mathbb{R}$ is a subset of $\mathcal{D}'(\mathbb{R})$. For any probability density function $P(x)$, $T_P \in \mathcal{D}'(\mathbb{R})$ is defined by

$$\langle T_P, \phi\rangle = \int_{\mathbb{R}} \phi(x)P(x)dx, \qquad \forall\phi \in \mathcal{D}(\mathbb{R}).$$

Example 4. The set of measures, $\mathcal{M}(\Omega)$, is a subset of $\mathcal{D}'(\Omega)$. For any $\mu \in \mathcal{M}(\Omega)$, $T_\mu \in \mathcal{D}'(\Omega)$ is defined by

$$\langle T_\mu, \phi\rangle = \int_\Omega \phi(x)d\mu, \qquad \forall\phi \in \mathcal{D}(\Omega).$$

Example 5. Dirac delta function.

As a typical example of measure, Delta function $\delta(x)$ is defined by

$$\langle \delta, \phi \rangle = \phi(0), \qquad \forall \phi \in \mathcal{D}(\Omega).$$

Remark 3.8. $\delta \notin L^1_{loc}$.

Proof. If not, there exists $f \in L^1_{loc}$ s.t. $\forall \phi \in C_0^\infty$,

$$\langle \delta, \phi \rangle = \int_{\mathbb{R}} f \phi dx.$$

Now we choose $\varphi_n(x) = \begin{cases} e^{\frac{1}{|nx|^2-1}}, & |nx| < 1 \\ 0, & |nx| \geq 1 \end{cases}$. Then we have on one hand

$$\langle \delta, \phi_n \rangle = \phi_n(0) = e^{-1}.$$

On the other hand, since $f \in L^1_{loc}$,

$$\int_{\mathbb{R}} f \varphi_n dx = \int_{|x| \leq \frac{1}{n}} f(x) e^{\frac{1}{|nx|^2-1}} dx \to 0, \text{ as } n \to \infty,$$

which is a contradiction. ∎

Definition 5. A sequence of distributions $\{T_n\}_{n=1}^\infty$ converges to T in the sense of distribution if

$$\langle T_n, \phi \rangle \to \langle T, \phi \rangle, \qquad \forall \phi \in \mathcal{D}(\Omega).$$

In the following we will show some sequences which converge to δ-function in the sense of distribution.

Example 6. Heat kernel $f_t(x) = \dfrac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$.

$\int_{\mathbb{R}} f_t(x) dx = 1$ and $\forall \phi \in C_0^\infty$,

$$\int_{\mathbb{R}} f_t(x) \phi(x) dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} \phi(2\sqrt{t} y) dy \to \phi(0),$$

by dominated convergence.

Example 7. $Q_n(x) = \begin{cases} \frac{n}{2}, & |nx| < 1 \\ 0, & |nx| \geq 1 \end{cases}$.

$\int_{\mathbb{R}} Q_n(x) dx = 1$ and $\forall \phi \in C_0^\infty$,

$$\int_{\mathbb{R}} Q_n(x) \phi(x) dx = \int_{-\frac{1}{n}}^{\frac{1}{n}} \frac{n}{2} \phi(x) dx \to \phi(0).$$

Example 8. Dirichlet kernel $D_n(x) = \dfrac{\sin(n + \frac{1}{2})x}{\sin \frac{x}{2}} = 1 + 2 \sum_{k=1}^n \cos kx$.

$\int_{\mathbb{R}} D_n(x)dx = 2\pi$ and $\forall\phi \in C_0^\infty$,

$$\int_{-\pi}^{\pi} D_n(x)\phi(x)dx = \int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}}\phi(x)dx \to 2\pi\phi(0),$$

which can be proved by using Riemann's lemma and the similar argument to those we have used in the proof of inverse Fourier transform for $L^1 \cap C^1$ functions.

Weak derivative of distributions The definition of weak derivative is enlightened by integration by parts. If $f \in C^1$, we have

$$\int_\Omega \partial_i f \phi dx = -\int_\Omega f \partial_i \phi dx, \quad \forall \phi \in C_0^\infty.$$

With the same idea, we can define the weak derivative for distributions.

Definition 6. $\forall T \in \mathcal{D}'(\Omega)$, $\partial_i T$ is defined by

$$\langle \partial_i T, \phi \rangle := -\langle T, \partial_i \phi \rangle, \quad \forall \phi \in \mathcal{D}(\Omega).$$

Since $-\partial_i\phi \in \mathcal{D}'(\Omega)$, it is easy to check that the right hand side defines a continuous linear functional on $\mathcal{D}'(\Omega)$. This new linear functional, as a distribution, is called the weak derivative of T. One can define higher order derivatives in the same way. Let α be a multi-index,

$$\langle \partial^\alpha T, \phi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \phi \rangle, \quad \forall \phi \in \mathcal{D}(\Omega).$$

Remark 3.9. According to this definition, we know that all distributions are infinitely weakly differentiable.

Example 9. The derivatives of δ-function.

$\forall \phi \in \mathcal{D}(\mathbb{R})$, we have

$$\langle \delta', \phi \rangle = -\langle \delta, \phi' \rangle = -\phi'(0),$$
$$\langle \delta^{(k)}, \phi \rangle = (-1)^k \langle \delta, \phi^{(k)} \rangle = (-1)^k \phi^{(k)}(0).$$

Example 10. The derivatives of Heaviside function $H = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$.

$\forall \phi \in \mathcal{D}(\mathbb{R})$, we have

$$\langle H', \phi \rangle = -\langle H, \phi' \rangle = -\int_0^\infty \phi'(x)dx = \phi(0) = \langle \delta, \phi \rangle.$$

Motivated by the change of variables, we can give the translation of distributions.

Definition 7.

$$\langle T(x-a), \phi(x) \rangle = \langle T(x), \phi(x+a) \rangle.$$

For example, $\delta_a(x) = \delta(x-a)$ is defined by

$$\langle \delta, \phi(x+a) \rangle = \phi(a).$$

3.1.3. *Tempered distribution and its Fourier transform.*

Definition 8. Schwartz class function $\mathcal{S}(\mathbb{R}^n)$, (rapidly decreasing function),

$$\mathcal{S}(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) | \sup_x |x^\gamma D^\alpha \phi| < +\infty, \forall \text{ multi-index } \alpha, \gamma\},$$

where the convergence of a sequence $\{\phi_j\} \subset \mathcal{S}(\mathbb{R}^n)$ to $\phi \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\sup_x |x^\gamma D^\alpha(\phi_j - \phi)| \to 0, \forall \text{ multi-index } \alpha, \gamma.$$

Theorem 3.5. *If $\phi \in \mathcal{S}(\mathbb{R}^n)$, then $\hat{\phi} \in \mathcal{S}(\mathbb{R}^n)$*

Proof. For all multi-index α, γ, by the properties of Fourier transform, we have

$$\left| k^\alpha D^\gamma \hat{\phi}(k) \right| = \left| \Big(D^\alpha \big((-x)^\gamma \phi(x) \big) \Big)^\wedge \right| \le \int_{\mathbb{R}^n} \left| D^\alpha \big((-x)^\gamma \phi(x) \big) \right| dx.$$

By taking the supremum in $k \in \mathbb{R}^n$, we have

$$\begin{aligned}
\sup_k |k^\alpha D^\gamma \hat{\phi}(k)| &\le \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |D^\alpha \big((-x)^\gamma \phi(x) \big)| dx \\
&\le \frac{1}{(2\pi)^{\frac{n}{2}}} C \sup_x (1 + |x|)^{n+1} |D^\alpha \big((-x)^\gamma \phi(x) \big)| < +\infty,
\end{aligned}$$

where $C = \displaystyle\int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+1}} dx.$

□

Definition 9. The dual space of $\mathcal{S}(\mathbb{R}^n)$ is called tempered distribution space, denoted by $\mathcal{S}'(\mathbb{R}^n)$.

Remark 3.10.

$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n), \qquad \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n).$$

Definition 10. $\forall T \in \mathcal{S}'(\mathbb{R}^n)$, its Fourier transform is defined by $\forall \phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle \hat{T}, \phi \rangle = \langle T, \hat{\phi} \rangle.$$

Example 11. The Fourier transform of Dirac delta function is a constant.

$\forall \phi \in \mathcal{S}(\mathbb{R}^n)$, by definition,

$$\langle \hat{\delta}, \phi \rangle = \langle \delta, \hat{\phi} \rangle = \hat{\phi}(0) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \phi(x) e^{-i0\cdot x} dx = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \phi(x) dx = \langle \frac{1}{(2\pi)^{\frac{n}{2}}}, \phi \rangle$$

This means that $\hat{\delta} = \dfrac{1}{(2\pi)^{\frac{n}{2}}}$ in the sense of distributions.

3.2. **Cauchy Problem.** The initial value problem of the heat equation is

$$u_t - \Delta u = f(x,t), \qquad x \in \mathbb{R}^n, t > 0, \tag{3.1}$$

$$u|_{t=0} = u_0(x). \tag{3.2}$$

3.2.1. *Solution formula of the Cauchy problem.* We will find the formal solution of Cauchy problem by Fourier transform. Taking the Fourier transform in x in equation (3.1) and as well as its initial value in (3.2) gives

$$\begin{aligned}\hat{u}_t + |k|^2\hat{u} = \hat{f}(k,t), &\qquad k \in \mathbb{R}^n, t > 0,\\ \hat{u}|_{t=0} = \hat{u}_0(k).&\end{aligned}$$

This ODE problem is easy to solve by Duhamel's formula

$$\hat{u}(k,t) = e^{-|k|^2 t}\hat{u}_0(k) + \int_0^t e^{-|k|^2(t-\tau)}\hat{f}(k,\tau)d\tau.$$

Now taking the inverse Fourier transform and using its property for convolutions, we have

$$\begin{aligned} u(x,t) &= \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} * u_0(x) + \int_0^t \frac{1}{(4\pi(t-\tau))^{\frac{n}{2}}} e^{-\frac{|x|^2}{4(t-\tau)}} * f(x,\tau)d\tau \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y)dy + \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-\tau))^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(t-\tau)}} f(y,\tau)dyd\tau. \end{aligned}$$

It can be seen from here that the function, known as **heat kernel**,

$$K(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, \tag{3.3}$$

is very important in getting the solution of the heat equation. Actually, the solution can be written in the following form

$$u(x,t) = K(x,t) * u_0 + \int_0^t K(x,t-\tau) * f(x,\tau)d\tau. \tag{3.4}$$

which is called **Poisson's formula**.

Theorem 3.6. *For homogeneous equation, i.e., $f \equiv 0$, if u_0 is a bounded function in $C(\mathbb{R})$, then $u(x,t)$ given by* (3.4) *is a bounded classical solution of* (3.1)-(3.2).

Proof. It is easy to see that $\forall t > 0$, $u(x,t) = K(x,t) * u_0(x)$ is infinitely differentiable both in x and t. Another fact is that

$$K_t - \Delta K = 0, \quad \forall t > 0.$$

These show that $u(x,t)$ is a solution of the equation for $t > 0$, i.e., $u_t - \triangle u = 0$ in $\mathbb{R}^n \times (0,\infty)$. Now we are left to check whether $u(x,t)$ satisfies the initial data. More precisely, it remains to show that,

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = u_0(x_0), \quad \forall x_0 \in \mathbb{R}^n.$$

By changing of variables in the integral, we have

$$\begin{aligned} u(x,t) &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y)dy \\ &= \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-|z|^2} u_0(x + 2\sqrt{t}z)dz. \end{aligned}$$

Due to the fact that u_0 is bounded, the above integral is uniformly convergent in x and t, which allows us to switch the order of the limit and the integral. Therefore

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = \frac{1}{\pi^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-|z|^2}u_0(x_0)dz = u_0(x_0).$$

□

Remark 3.11. Some basic properties of the solution $u(x,t) = \dfrac{1}{(4\pi t)^{\frac{n}{2}}}\displaystyle\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}u_0(y)dy$ can be obtained directly from this formula.

(1) If u_0 is periodic (or odd, or even) in x, so is $u(x,t)$.
(2) Infinite speed of propagation. If $u_0(x) \geq 0$ has support in a small domain, say $\Omega_0 \subset \mathbb{R}^n$, $u(x,t)$ is positive everywhere in $\mathbb{R}^n$.
(3) Infinite differentiability of the solution $u(x,t)$ for $t > 0$.

Next we consider the inhomogeneous equation with homogeneous initial value

$$\begin{aligned} u_t - \Delta u = f, &\quad \text{in } \mathbb{R}^n \times (0,+\infty), \\ u|_{t=0} = 0. & \end{aligned} \tag{3.5}$$

The solution is given by

$$\begin{aligned} u(x,t) &= \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s)f(y,s)dyds \\ &= \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}}f(y,s)dyds. \end{aligned} \tag{3.6}$$

Theorem 3.7. *If $f \in C^{2,1}(\mathbb{R}^n\times[0,\infty))$ and has compact support, then the function u given by* (3.6) *is a function in $C^{2,1}(\mathbb{R}^n \times [0,\infty))$, furthermore, it is a solution of* (3.5).

Proof. By the regularity of f, we have

$$\begin{aligned} &(\partial_t - \Delta)u(x,t) \\ =\ & \int_0^t \int_{\mathbb{R}^n} K(y,s)(f_t-\Delta f)(x-y,t-s)dyds + \int_{\mathbb{R}^n} K(y,t)f(x-y,0)dy \\ =\ & \int_0^\varepsilon \int_{\mathbb{R}^n} K(y,s)(f_t-\Delta f)(x-y,t-s) \\ & + \int_\varepsilon^t \int_{\mathbb{R}^n} K(y,s)(f_t-\Delta f)(x-y,t-s) \\ & + \int_{\mathbb{R}^n} K(y,t)f(x-y,0)dy =: J_\varepsilon + I_\varepsilon + L. \end{aligned}$$

We deal with the right hand side term by term,

$$|J_\varepsilon| \leq (\|f_t\|_{L^\infty} + \|D^2 f\|_{L^\infty})\int_0^\varepsilon \int_{\mathbb{R}^n} K(y,s)dyds \leq \varepsilon C.$$

$$\begin{aligned} I_\varepsilon &= \int_\varepsilon^t \int_{\mathbb{R}^n} K(y,s)(f_t - \Delta f)(x-y,t-s)dyds \\ &= \int_\varepsilon^t \int_{\mathbb{R}^n} (\partial_s - \Delta_y)K(y,s)f(x-y,t-s)dyds + \int_{\mathbb{R}^n} K(y,\varepsilon)f(x-y,t-\varepsilon)dy \\ &\qquad - \int_{\mathbb{R}^n} K(y,t)f(x-y,0)dy \\ &= \int_{\mathbb{R}^n} K(y,\varepsilon)f(x-y,t-\varepsilon)dy - L. \end{aligned}$$

Therefore we have

$$u_t - \Delta u = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} K(y,\varepsilon)f(x-y,t-\varepsilon)dy = f(x,t), \quad \forall t>0,$$

and

$$|u(x,t)| = \Big|\int_0^t \int_{\mathbb{R}^n} K(y,s)f(x-y,t-s)dyds\Big| \le t\|f\|_{L^\infty} \to 0, \quad \text{as } t\to 0,$$

which means that the solution also satisfies the initial condition. □

Remark 3.12. By superposition principle for linear equations, under the conditions of theorem 3.6 and 3.7, we claim

$$u(x,t) = \int_{\mathbb{R}^n} K(x-y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s)f(y,s)dyds$$

is the solution of

$$u_t - \Delta u = f, \quad u|_{t=0} = u_0.$$

3.2.2. *Fundamental solution.* We first give some motivations in defining fundamental solutions. Formally, the right hand side function $f(x,t)$, the heat source, can be represented by

$$f(x,t) = \int_0^\infty \int_{\mathbb{R}^n} \delta_{(\xi,\tau)}(x,t)f(\xi,\tau)d\xi d\tau, \quad \forall x\in\mathbb{R}^n, t>0,$$

which means that $f(x,t)$ can be treated as a summation of $\delta_{(\xi,\tau)}(x,t)f(\xi,\tau)d\xi d\tau$, the point heat source. Therefore, we can expect that if $K(x,t;\xi,\tau)$ is the solution of $u_t - \Delta_x u = \delta_{(\xi,\tau)}(x,t)$, then $K(x-\xi,t-\tau)f(\xi,\tau)d\xi d\tau$ is the solution with point heat source. Thus in the case of general heat source $f(x,t)$, the solution can be written into a superposition of point heat sources,

$$u(x,t) = \int_0^\infty \int_{\mathbb{R}^n} K(x-\xi,t-\tau)f(\xi,\tau)d\xi d\tau.$$

Basically, the fundamental solution of heat equation is to find the temperature distribution with a point heat source at (ξ,τ).

Definition 11. $K(x,t;\xi,\tau) = K(x-\xi,t-\tau)$ is called the **fundamental solution** of heat equation.

Let $Q = \mathbb{R}^n \times (0, \infty)$. $\forall (x,t) \in Q$. $K(x,t;\xi,\tau)$ is a solution (in the sense of distribution) of the following Cauchy problem

$$\begin{aligned} u_t - \Delta_x u &= \delta_{(\xi,\tau)}(x,t), \\ u|_{t=0} &= 0. \end{aligned}$$

Remark 3.13. We also know that $K(x,t;\xi,\tau)$ is a solution of

$$\begin{aligned} &u_t - \Delta_x u = 0, \qquad \forall x \in \mathbb{R}^n, t \geq \tau, \\ &u|_{t=\tau} = \delta_\xi(x). \end{aligned}$$

Some properties of the fundamental solution

(1) $K(x,t;\xi,\tau) > 0$ for $t > \tau$.
(2) $K(x,t;\xi,\tau) = K(\xi,t;x,\tau)$.
(3) $\forall x \in \mathbb{R}^n$, $t > \tau$,

$$\int_{\mathbb{R}^n} K(x,t;\xi,\tau)d\xi = 1.$$

(4) $\forall x, \xi \in \mathbb{R}^n$, $t > \tau$,

$$\begin{aligned} (\partial_t - \Delta_x)K(x,t;\xi,\tau) = 0 \\ (\partial_\tau + \Delta_\xi)K(x,t;\xi,\tau) = 0 \end{aligned}$$

(5) If $\varphi(x)$ is a bounded continuous function in $\mathbb{R}^n$, then

$$\lim_{t \to 0+} \int_{\mathbb{R}^n} K(x,t;\xi,0)\varphi(\xi)d\xi = \varphi(x).$$

(6) $K(x,t;\xi,\tau)$ is infinitely differentiable and $\exists M > 0$ s.t. in the case of $t > \tau$,

$$|K(x,t;\xi,\tau)| \leq \frac{M}{(t-\tau)^{\frac{n}{2}}}.$$

Remark 3.14. There is another derivation of fundamental solution instead of using Fourier transform. One can check this method in Evan's book.

3.3. **Half space problem and its Green's function.** The main purpose of this section is to give a first insight on how to build up a Green's function on general problem.

Consider the problem

$$\begin{aligned} u_t - u_{xx} &= 0, \quad x \in (0, +\infty), t > 0, \\ u|_{t=0} &= \varphi, \quad x \in [0, +\infty), \\ u|_{x=0} &= 0, \quad t > 0. \end{aligned} \tag{3.7}$$

We want to find a function $G(x,t,\xi,0)$ such that the solution of (3.7) can be represented by

$$u(x,t) = \int_0^\infty G(x,t;\xi,0)\varphi(\xi)d\xi.$$

The important thing here is that we must make sure the solution satisfies the boundary condition $u|_{x=0} = 0$. $\forall \xi \in (0, +\infty)$, if the initial data is $\delta_\xi(x)$, we need to find the odd extension of it, i.e., $-\delta_{-\xi}(x)$, to balance the boundary condition. Now we can choose the initial data as

$$\delta_\xi(x) - \delta_{-\xi}(x),$$

and solve the Cauchy problem with this initial data. Since the problem is linear, our the solution is exactly

$$K(x, t; \xi, 0) - K(x, t; -\xi, 0).$$

Thus the Green's function for half space problem (3.7) can be written as

$$G(x, t; \xi, 0) = K(x, t; \xi, 0) - K(x, t; -\xi, 0),$$

and the solution of (3.7) is expected to be $u(x, t) = \displaystyle\int_0^\infty G(x, t; \xi, 0)\varphi(\xi)d\xi$.

Theorem 3.8. *φ is a bounded smooth function on $(0, +\infty)$ and $\varphi(0) = 0$, $u(x, t) = \displaystyle\int_0^\infty G(x, t; \xi, 0)\varphi(\xi)d\xi$ is the solution of (3.7).*

The proof of this theorem is left to the reader.

Remark 3.15. For inhomogeneous problem

$$\begin{aligned} u_t - u_{xx} &= f, \quad x \in (0, +\infty), t > 0, \\ u|_{t=0} &= \varphi, \quad x \in [0, +\infty), \\ u|_{x=0} &= 0, \quad t > 0. \end{aligned}$$

The formal solution is

$$u(x, t) = \int_0^\infty G(x, t; \xi, 0)\varphi(\xi)d\xi + \int_0^t d\tau \int_0^\infty G(x, t; \xi, \tau)f(\xi, \tau)d\xi.$$

Remark 3.16. Similarly, one can find the Green's function for half space problem with homogeneous Neumann boundary condition.

3.4. **Initial boundary value problem.** The initial boundary value (with homogeneous Dirichlet boundary condition) problem for heat equation in 1-d space variable is

$$\begin{aligned} u_t - u_{xx} &= f, \quad x \in (0, 1), t > 0, \\ u|_{t=0} &= \varphi, \quad x \in (0, 1), \\ u|_{x=0} = u|_{x=1} &= 0, \quad t > 0. \end{aligned} \tag{3.8}$$

The method of separation of variables is easy to be applied here.

3.4.1. *Separation of variables.* First by solving the eigenvalue problem

$$\begin{aligned} X'' + \lambda X = 0, &\qquad x \in (0,1), \\ X(0) = X(1) = 0, & \end{aligned}$$

we have

$$\lambda_n = (n\pi)^2, \quad X_n = \sin n\pi x.$$

Then if the solution $u(x,t)$ has form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin n\pi x,$$

$T_n(t)$ will solve the initial value problem of an ODE

$$\begin{aligned} T_n' + (n\pi)^2 T_n &= f_n(t), \\ T_n(0) &= \varphi_n, \end{aligned}$$

where

$$f_n(t) = 2\int_0^1 f(x,t)\sin n\pi x dx, \qquad \varphi_n = 2\int_0^1 \varphi(x)\sin n\pi x dx.$$

This ODE problem has a solution

$$T_n(t) = e^{-(n\pi)^2 t}\varphi_n + \int_0^t e^{-(n\pi)^2(t-\tau)} f_n(\tau)d\tau, \quad n = 1, 2, \cdots.$$

Thus the formal solution for problem (3.8) can be written as

$$u(x,t) = \sum_{n=1}^{\infty} \sin n\pi x \Big(e^{-(n\pi)^2 t}\varphi_n + \int_0^t e^{-(n\pi)^2(t-\tau)} f_n(\tau)d\tau \Big). \tag{3.9}$$

A natural question to ask is under what condition is (3.9) a $C^{2,1}((0,1)\times(0,\infty))$ solution. Here we give a result with sufficient assumption on the given data.

Theorem 3.9. *If $\varphi(x) \in C[0,1]$ and $f, f_t, f_{xxt} \in C^{2,1}((0,1)\times(0,\infty))$ and have compact support in x, then the formula given by (3.9) is in $C^{2,1}((0,1)\times(0,\infty))$ and is a solution of the problem (3.8).*

Proof. The uniform convergence of the formula $u(x,t)$ is omitted here. The more difficult steps are shown in the following, i.e., to prove that u_{xx} and u_t also converge uniformly in x, t.

If we take twice derivatives in x or derivative in t in the formula (3.9) on each term in the summation, we will have $(n\pi)^2$. Therefore, in order to prove that it is the solution, we only need to show that $\forall t > 0$, the following series converge uniformly in x and t, i.e.,

$$\begin{aligned} &\sum_{n=1}^{\infty} (n\pi)^2 \sin n\pi x e^{-(n\pi)^2 t}\varphi_n; \\ &\sum_{n=1}^{\infty} (n\pi)^2 \sin n\pi x \int_0^t e^{-(n\pi)^2(t-\tau)} f_n(\tau)d\tau. \end{aligned}$$

The convergence of the first series is obvious because of the exponential term $e^{-(n\pi)^2 t}$. To deal with the second series, we need to "produce" a factor $\dfrac{1}{(n\pi)^2}$ by using integration by parts on the integrals involved in the formula.

$$\begin{aligned}
&\int_0^t e^{-(n\pi)^2(t-\tau)} f_n(\tau)d\tau \\
= \;& \frac{f_n(t)}{(n\pi)^2} - \frac{e^{-(n\pi)^2 t} f_n(0)}{(n\pi)^2} - \int_0^t \frac{1}{(n\pi)^2} e^{-(n\pi)^2(t-\tau)} f_n'(\tau)d\tau.
\end{aligned}$$

Therefore, the second series can be rewritten into

$$\begin{aligned}
&\sum_{n=1}^{\infty} (n\pi)^2 \sin n\pi x \int_0^t e^{-(n\pi)^2(t-\tau)} f_n(\tau)d\tau \\
= \;& \sum_{n=1}^{\infty} \sin n\pi x \Big(f_n(t) - e^{-(n\pi)^2 t} f_n(0) - \int_0^t e^{-(n\pi)^2(t-\tau)} f_n'(\tau)d\tau \\
= \;& f(x,t) - f_n(0) \sum_{n=1}^{\infty} \sin n\pi x e^{-(n\pi)^2 t} - \sum_{n=1}^{\infty} \sin n\pi x \int_0^t e^{-(n\pi)^2(t-\tau)} f_n'(\tau)d\tau,
\end{aligned}$$

where the convergence of the last series can be done by using exactly the same method as for the convergence of $u(x,t)$ itself. Namely, with the assumption that f has compact support, we do integration by parts twice,

$$\begin{aligned}
f_n'(\tau) &= 2\int_0^1 f_t(x,\tau) \sin n\pi x dx \\
&= -\frac{2}{(n\pi)^2} \int_0^1 f_{xxt}(x,\tau) \sin n\pi x dx =: -\frac{2B_n(\tau)}{(n\pi)^2}.
\end{aligned}$$

Due to the fact that $\sum_{n=1}^{\infty} B_n^2 < \infty$, we have

$$\begin{aligned}
&\Big| - \sum_{n=1}^{\infty} \sin n\pi x \int_0^t e^{-(n\pi)^2(t-\tau)} f_n'(\tau)d\tau \Big| \\
\le \;& \sum_{n=1}^{\infty} \frac{2|\sin n\pi x|}{(n\pi)^2} \int_0^t e^{-(n\pi)^2(t-\tau)} |B_n(\tau)|d\tau \\
\le \;& \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2} \int_0^t (1 + B_n^2(\tau))d\tau < \infty.
\end{aligned}$$

□

Basic properties of the solution of heat (or more generally, parabolic) equation, In the case of no external source, i.e., $f = 0$, the solution has "Infinitely differentiablility inside of the domain". It is mainly due to that $e^{-(n\pi)^2 t}$ decays exponentially in n for any positive time t. More precisely, $\forall (x,t) \in (0,1) \times (0,\infty)$, for any nonnegative integer

k, l, the solution given by (3.9) is $(k+l)$ times differentiable at (x,t). For example, we know that

$$\frac{\partial^{k+l}u(x,t)}{\partial x^k \partial t^l} = \sum_{n=1}^{\infty}(-1)^l(n\pi)^{k+2l}\varphi_n e^{-(n\pi)^2 t}\sin(n\pi x + \frac{k\pi}{2})$$

converges uniformly in x and positive t.

3.4.2. *Energy estimates.* We will give the energy estimate for initial boundary value problem of heat equation in multi-dimension. Ω is a bounded open subset of $\mathbb{R}^n$. Let $Q_T = \Omega \times (0,T)$.

$$\begin{aligned} u_t - \Delta u &= f, \quad (x,t) \in Q_T, \\ u|_{t=0} &= \varphi, \quad x \in \Omega, \\ u|_{\partial\Omega} &= 0, \quad t > 0. \end{aligned} \tag{3.10}$$

Theorem 3.10. *If $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ is a solution of problem* (3.10)*, then $\exists M > 0$ depends only on T, s.t.*

$$\sup_{0\le t\le T}\|u(\cdot,t)\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(Q_T)} \le M(\|\varphi\|_{L^2(\Omega)} + \|f\|_{L^2(Q_T)}). \tag{3.11}$$

Proof. By multiplying the equation by u, integrating it over Q_t and integrating by parts in x, we have

$$\frac{1}{2}\int_\Omega u^2 dx - \frac{1}{2}\int_\Omega \varphi^2 dx + \int_0^t\int_\Omega |\nabla u|^2 dxdt = \int_0^t\int_\Omega f u dxdt.$$

Young's inequality gives

$$\int_\Omega u^2 dx + 2\int_0^t\int_\Omega |\nabla u|^2 dxdt \le \int_0^t\int_\Omega u^2 dxdt + \int_\Omega \varphi^2 dx + \int_0^t\int_\Omega f^2 dxdt$$

Then (3.11) can be obtained directly from Gronwall's inequality.

□

Remark 3.17. The discussion on uniqueness and stability of the solution by energy estimates is similar to what we have done for the wave equation.

Remark 3.18. For homogeneous Neumann boundary condition $u \cdot \gamma|_{\partial\Omega} = 0$, where γ is the unit outer normal vector of $\partial\Omega$, the energy estimate can be done similarly.

Remark 3.19. For inhomogeneous boundary condition, i.e., $u|_{\partial\Omega} = \psi_D$, one can try to homogenize it or just use $u - \psi_D$ as test function.

3.5. **Maximum principle.** In this subsection, we introduce one of the basic tools, the so-called maximum principle, for second order elliptic or parabolic equation. It is easy to get the main idea from handling the heat equation.

In the following, we assume that Ω is a bounded open subset of $\mathbb{R}^n$. Let $Q_T = \Omega \times (0,T]$, the parabolic boundary of Q_T be $\partial_p Q_T = \Omega \times \{t = 0\} \cup \partial\Omega \times (0,T]$ and $Lu = u_t - \Delta u$.

3.5.1. *Weak maximum principle.*

Theorem 3.11. *If $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ and $Lu \leq 0$ in Q_T, then the maximum of u in $\bar{Q}_T$ must be achieved on $\partial_p Q_T$, i.e.,*

$$\max_{\bar{Q}_T} u(x,t) = \max_{\partial_p Q_T} u. \tag{3.12}$$

Proof. We first assume $Lu < 0$ in Q_T. If (3.12) is not true, which means $\exists (x_0, t_0) \in Q_T$ such that

$$u(x_0, t_0) = \max_{\bar{Q}_T} u(x,t),$$

then we know that $\nabla u(x_0,t_0) = 0$, $\Delta u(x_0,t_0) \leq 0$ and $u_t(x_0,t_0) \geq 0$. Therefore,

$$f(x_0,t_0) = Lu(x_0,t_0) \geq 0,$$

which is a contradiction with the assumption $Lu < 0$.

If $Lu \leq 0$, we introduce an auxiliary function

$$v(x,t) = u(x,t) - \varepsilon t, \quad \forall 0 < \varepsilon << 1.$$

Then the application of the heat operator on v gives

$$Lv = Lu - \varepsilon = f - \varepsilon < 0.$$

By the conclusion we obtained above, we have

$$\max_{\bar{Q}_T} v = \max_{\partial_p Q_T} v.$$

Going back to the variable u,

$$\begin{aligned} \max_{\bar{Q}_T} u(x,t) &= \max_{\bar{Q}_T} (v + \varepsilon t) \leq \max_{\bar{Q}_T} v + \varepsilon T \\ &\leq \max_{\partial_p Q_T} v + \varepsilon T = \max_{\partial_p Q_T} (u - \varepsilon t) + \varepsilon T \\ &\leq \max_{\partial_p Q_T} u + \varepsilon T. \end{aligned}$$

In the end, taking $\varepsilon \to 0$ finishes the proof, i.e., we have (3.12). □

By the same discussion or letting $v = -u$, we will have

Corollary 3.1. *If $Lu \geq 0$, then*

$$\min_{\bar{Q}_T} u(x,t) = \min_{\partial_p Q_T} u.$$

A combination of the above two results gives

Corollary 3.2. *If $Lu = 0$, then both the maximum and the minimum of u are achieved on the parabolic boundary.*

Now we will have the very useful tool, the **comparison principle**, as a corollary

Corollary 3.3. *If $u, v \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$, $Lu \leq Lv$ and $u|_{\partial_p Q_T} \leq v|_{\partial_p Q_T}$, then*

$$u(x,t) \leq v(x,t), \quad \text{in } \bar{Q}_T.$$

3.5.2. *Dirichlet boundary value problem.* The initial boundary value problem of heat equation with Dirichlet boundary condition is

$$\begin{aligned} u_t - \Delta u &= f(x,t), \quad \text{in } Q_T, \\ u|_{t=0} &= \varphi(x), \\ u|_{\partial\Omega} &= g(x,t). \end{aligned} \tag{3.13}$$

Theorem 3.12. *If $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ is a solution of (3.13), then*

$$\max_{\bar{Q}_T} |u| \le FT + B, \tag{3.14}$$

where $F = \max_{\bar{Q}_T} |f|$, $B = \max\{\max_\Omega |\varphi|, \max_{\partial\Omega\times[0,T]} |g|\}$.

Proof. The proof will be obtained by using comparison principle. We introduce auxiliary function $w(x,t) = Ft + B \pm u(x,t)$. It is easy to check that

$$\begin{aligned} Lw &= F \pm f \ge 0, \\ w|_{\partial_p Q_T} &\ge Ft + B \pm g|_{\partial_p Q_T} \ge 0. \end{aligned}$$

By comparison principle, Corollary 3.3, we have $w(x,t) \ge 0$ in Q_T, which implies

$$|u| \le FT + B, \quad \text{in } Q_T.$$

□

This maximum estimate can be used to prove the uniqueness and stability of classical solutions.

Corollary 3.4. *$C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ solution of (3.13) is unique.*

Corollary 3.5. *$C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ solution of (3.13) is stable in the following sense. If $u_1, u_2 \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ are solutions separately with data φ_1, f_1, g_1 and φ_2, f_2, g_2, then*

$$\max_{\bar{Q}_T} |u_1 - u_2| \le \|f_1 - f_2\|_\infty T + \|\varphi_1 - \varphi_2\|_\infty + \|g_1 - g_2\|_\infty.$$

3.6. **Long time behavior of the solution.** In this subsection, we give further discussion of the long time behavior of the solution to heat equation based on energy or entropy estimates.

3.6.1. *Poincaré's inequalities.* First, we introduce Poincaré type of inequalities. Ω is a connected open bounded domain in $\mathbb{R}^n$

Lemma 3.1. *(Poincaré's inequality for $C_0^1(\Omega)$ functions) For all $u \in C_0^1(\Omega)$, it holds*

$$\|u\|_{L^2(\Omega)} \le 2d\|\nabla u\|_{L^2(\Omega)},$$

where $d =$ diam Ω.

Proof. For all $u \in C_0^1(\Omega)$, without loss of generality we assume

$$\Omega \subset \{x | 0 \le x_i \le 2d, 1 \le i \le n\} = \bar{Q}.$$

Let $\tilde{u} = \begin{cases} u, & x \in \bar{\Omega} \\ 0, & x \in \bar{Q} \backslash \bar{\Omega} \end{cases}$. It is obvious that $\tilde{u}$ is a piecewise C^1 function, and

$$\tilde{u}|_{\partial \bar{Q}} = 0.$$

By Newton-Leibnitz formula

$$\tilde{u}(x_1, x_2, \cdots, x_n) = \int_0^{x_1} \frac{\partial \tilde{u}}{\partial x_1} dx_1,$$

then

$$\tilde{u}^2 = \Big(\int_0^{x_1} \frac{\partial \tilde{u}}{\partial x_1} dx_1 \Big)^2 \le x_1 \int_0^{x_1} \Big(\frac{\partial \tilde{u}}{\partial x_1} \Big)^2 dx_1 \le 2d \int_0^{2d} \left| \frac{\partial \tilde{u}}{\partial x_1} \right|^2 dx_1.$$

By taking integration over Q, we have

$$\int_Q \tilde{u}^2 dx \le 2d \int_Q \int_0^{2d} \left| \frac{\partial \tilde{u}}{\partial x_1} \right|^2 dx_1 dx \le 4d^2 \int_Q |\nabla \tilde{u}|^2 dx.$$

Therefore,

$$\|u\|_{L^2(\Omega)} \le 2d \|\nabla u\|_{L^2(\Omega)}.$$

□

Lemma 3.2. *(Poincaré's inequality for $C^1(\Omega)$ functions) For all $u \in C^1(\Omega)$, it holds*

$$\|u - \bar{u}\|_{L^2(\Omega)} \le C(\Omega) \|\nabla u\|_{L^2(\Omega)},$$

where $\bar{u} = \frac{1}{|\Omega|} \int_\Omega u dx$ and C is a constant depending on Ω.

Proof. We only prove the case $\Omega = (0, 1)$. The more general case can be proved by using Sobolev embedding theorem. Since u is a continuous function, there exists $x_0 \in (0, 1)$ such that $u(x_0) = \bar{u}$. Then by Newton-Leibnitz formula,

$$(u(x) - \bar{u})^2 = \Big(\int_{x_0}^x u'(y) dy \Big)^2 \le (x - x_0) \int_{x_0}^x |u'(y)|^2 dy.$$

Therefore the inequality is obtained by taking integration over $(0, 1)$. □

A more general version is given in the following. We will skip the proof here.

Lemma 3.3. *(Gaussian Poincaré's inequality for $C^1(\mathbb{R}^n)$ functions with measure $d\mu = G dx$) For all $u \in C^1(\mathbb{R}^n)$, it holds*

$$\|u - \bar{u}\|_{L^2(\mathbb{R}^n, d\mu)} \le \|\nabla u\|_{L^2(\mathbb{R}^n, d\mu)},$$

where $\bar{u} = \int_{\mathbb{R}^n} u d\mu$ and $d\mu = G(x) dx = (2\pi)^{-\frac{n}{2}} e^{-\frac{x^2}{2}} dx$.

3.6.2. *Heat equation with Dirichlet boundary condition.* Next we use these two inequalities to prove the long time behavior of the solution to heat equation.

Theorem 3.13. *If $u \in C^{2,1}(\Omega \times (0,\infty)) \cap C(\bar{\Omega} \times [0,\infty))$ is a solution of the problem* (3.10) *with $f = 0$, then $\exists C(\Omega)$ such that*

$$\|u(x,t)\|_{L^2(\Omega)} \le \|\varphi\|_{L^2(\Omega)} e^{-Ct},$$

furthermore for any fixed x, $u(x,t) \to 0$ as $t \to \infty$.

Proof. Use u as test function in the heat equation and notice that $u|_{\partial\Omega} = 0$, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(x,t)dx + \int_\Omega |\nabla u|^2 dx = 0.$$

With the help of lemma 3.1, we have

$$\frac{d}{dt}\int_\Omega u^2(x,t)dx + \frac{1}{d}\int_\Omega |u|^2 dx \le 0.$$

Therefore,

$$\int_\Omega u^2(x,t)dx \le e^{-\frac{t}{d}}\int_\Omega \varphi^2.$$

□

3.6.3. *Heat equation with Neumann boundary condition***.*

Theorem 3.14. *If $u \in C^{2,1}(\Omega \times (0,\infty)) \cap C(\bar{\Omega} \times [0,\infty))$ is a solution of the problem*

$$u_t - \Delta u = 0, \quad \nabla u \cdot \gamma|_\Omega = 0, \quad u|_{t=0} = \varphi(x),$$

then $\exists C(\Omega)$ such that

$$\|u(x,t) - \bar{\varphi}\|_{L^2(\Omega)} \le \|\varphi - \bar{\varphi}\|_{L^2(\Omega)} e^{-Ct},$$

furthermore for any fixed x, $u(x,t) \to \bar{\varphi}$ as $t \to \infty$.

Proof. A very important fact of the heat equation with homogeneous boundary condition is that

$$\int_\Omega u(x,t)dx \equiv \int_\Omega \varphi(x)dx,$$

which can be obtained immediately by integrating the equation over Ω, where the diffusion term disappears because of the Neumann boundary condition. Thus $\bar{u} \equiv \bar{\varphi}$, which is independent of time.

Use $u - \bar{u}$ as test function in the heat equation and notice that u satisfies the homogeneous Neumann boundary condition, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u(x,t) - \bar{u})^2 dx + \int_\Omega |\nabla u|^2 dx = 0.$$

With the help of lemma 3.2, we have

$$\frac{d}{dt}\int_\Omega (u(x,t) - \bar{u})^2 dx + \frac{2}{C}\int_\Omega (u(x,t) - \bar{u})^2 dx \le 0.$$

Therefore,

$$\int_\Omega (u(x,t)-\bar{u})^2 dx \le e^{-\frac{2t}{C}} \int_\Omega (\varphi-\bar{\varphi})^2.$$

$\square$

3.6.4. *Heat equation, Cauchy problem.* In this part we concentrate on the Cauchy problem of heat equation and by using different scaling of the variable to observe more details of the solution behavior.

Let $u(y,\tau) \in C^{2,1}$ be the solution of

$$u_\tau = \frac{1}{2}\Delta_y u,$$

and with the initial data u_0 such that $\int u_0 dx = 1$. Note that the fundamental solution is $K(y,\tau) = C\frac{1}{\tau^{\frac{n}{2}}} e^{-\frac{y^2}{2\tau}}$, we do a new space-time rescaling, i.e., $t = \log(1+\tau)$, $x = \frac{y}{(1+\tau)^{\frac{1}{2}}}$ and let

$$u(y,\tau) = v(x,t)(1+\tau)^{-\frac{n}{2}}$$

to perserve the total mass of the solution, i.e., $\int_{\mathbb{R}^n} v(x,t)dx = \int_{\mathbb{R}^n} u(y,t)dy$. v satisfies the following so called Fokker-Planck equation

$$v_t = \frac{1}{2}\Delta_x v + \frac{1}{2}\nabla\cdot(xv) = \frac{1}{2}\nabla\cdot(v(\nabla \ln v + x)).$$

It is obvious that $G = \pi^{-n/2} e^{-\frac{x^2}{2}}$ is a stationary solution of it.

If we take further $w = \frac{v}{G}$, w satisfies the following so called Ornstein-Uhlenbeck equation (a symmetrically weighted heat equation)

$$w_t = \frac{1}{2}G^{-1}\nabla\cdot(G\nabla w).$$

Notice that all of the transformation keeps the total mass, i.e.,

$$\int_{\mathbb{R}^n} w d\mu = \int_{\mathbb{R}^n} v(x,t)dx = \int_{\mathbb{R}^n} u(y,t)dy = 1.$$

Theorem 3.15 (Energy estimate)**.** *Let $u(y,\tau)$ be a solution of the heat equation and initial data satisfies $\int_{\mathbb{R}^n} |u_0 - G|^2 dx < \infty$, then there exists a constant C such that*

$$\int_{\mathbb{R}^n} \left| u(y,\tau) - G\Big(\frac{y}{(1+\tau)^{\frac{1}{2}}}\Big)(1+\tau)^{-\frac{n}{2}}\right|^2 \frac{1}{G(\frac{y}{(1+\tau)^{\frac{1}{2}}})} dy \le \frac{C}{(1+\tau)^{\frac{n}{2}+1}}.$$

Proof. We start with an energy estimate for w. Let $\mathcal{F}(w(t)) = \int_{\mathbb{R}^n} |w-1|^2 d\mu$, then

$$\frac{d\mathcal{F}(w(t))}{dt} = -\int_{\mathbb{R}^n} |\nabla w|^2 G(x) dx =: -\mathcal{D}(w(t)).$$

By the Gaussian Poincaré's inequality in lemma 3.3 with measure $d\mu = G(x)dx$, we have

$$\int_{\mathbb{R}^n} |w-1|^2 d\mu \leq \int_{\mathbb{R}^n} |\nabla w|^2 d\mu.$$

Therefore

$$\int_{\mathbb{R}^n} |w-1|^2 d\mu \leq e^{-t}\int_{\mathbb{R}^n} |w_0-1|^2 d\mu = e^{-t}\int_{\mathbb{R}^n} |u_0-G|^2 dx. \tag{3.15}$$

The result is obtained by changing back to the original variable, $u(y,\tau) = v(x,t)(1+\tau)^{-\frac{n}{2}} = w(x,t)G(x,t)(1+\tau)^{-\frac{n}{2}}$, namely,

$$\begin{aligned}
&\int_{\mathbb{R}^n} |w(t)-1|^2 d\mu = \int_{\mathbb{R}^n} |v(x,t)-G(x)|^2 \frac{1}{G(x)} dx \\
= \ & \int_{\mathbb{R}^n} \Big| u(y,\tau)(1+\tau)^{\frac{n}{2}} - G\Big(\frac{y}{(1+\tau)^{\frac{1}{2}}}\Big)\Big|^2 \frac{1}{G(\frac{y}{(1+\tau)^{\frac{1}{2}}})} \frac{dy}{(1+\tau)^{\frac{n}{2}}} \\
= \ & (1+\tau)^{\frac{n}{2}} \int_{\mathbb{R}^n} \Big| u(y,\tau) - G\Big(\frac{y}{(1+\tau)^{\frac{1}{2}}}\Big)(1+\tau)^{-\frac{n}{2}}\Big|^2 \frac{1}{G(\frac{y}{(1+\tau)^{\frac{1}{2}}})} dy
\end{aligned}$$

and the right hand side with variable τ is

$$e^{-t}\int_{\mathbb{R}^n} |u_0-G|^2 dx = C\frac{1}{1+\tau},$$

which completes the proof. □

Remark 3.20. From (3.15), instead of changing back to the variable immediately, we can use the estimate from weighted heat equation

$$\|w-1\|_\infty \leq C\|w-1\|_{L^2}.$$

We can get

$$\|w-1\|_\infty \leq Ke^{-t}.$$

Therefore, we get a point-wise estimate,

$$(1-\frac{C}{1+\tau})(1+\tau)^{\frac{n}{2}} G(\frac{y}{(1+\tau)^{\frac{1}{2}}}) \leq u(y,\tau) \leq (1+\frac{C}{1+\tau})(1+\tau)^{\frac{n}{2}} G(\frac{y}{(1+\tau)^{\frac{1}{2}}}).$$

Remark 3.21. One can also proceed to do the entropy estimate. We start with an entropy estimate for v. Let

$$\mathcal{E}(v(t)) = \int_{\mathbb{R}^n} \frac{v}{G}\log\frac{v}{G} d\mu = \int_{\mathbb{R}^n} v\log v dx + \frac{1}{2}\int_{\mathbb{R}^n} x^2 v + C,$$

then

$$\frac{d\mathcal{E}(v(t))}{dt} = -\int_{\mathbb{R}^n} v\Big|\frac{\nabla v}{v} + x\Big|^2 dx = -\int_{\mathbb{R}^n} v\Big|\nabla\log(\frac{v}{G})\Big|^2 dx =: -\mathcal{I}(v(t)).$$

By using the logarithmic Sobolev inequality, i.e. $\displaystyle\int f^2\log f dx \leq \int |\nabla f|^2 dx$, we have

$$\mathcal{E} \leq \frac{1}{2}\mathcal{I}.$$

Therefore,

$$\frac{d\mathcal{E}}{dt} + 2\mathcal{E} \leq 0 \Longrightarrow \mathcal{E}(t) \leq e^{-2t}\mathcal{E}(0).$$

3.7. **Problems.**

(1) Find the formal solution of the following problem by Fourier transform

$$\begin{aligned} &i\partial_t u + \Delta u = 0, \quad (x,t) \in \mathbb{R}^n \times (0,+\infty), \\ &u|_{t=0} = g(x), \qquad x \in \mathbb{R}^n. \end{aligned}$$

(2) Find the formal solution of the following problems

(a)

$$\begin{aligned} u_t - \Delta u + 2u = f(x,t), &\quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u|_{t=0} = \varphi(x), &\quad x \in \mathbb{R}^n. \end{aligned}$$

(b)

$$\begin{aligned} u_t - u_{xx} + xu = 0, &\quad (x,t) \in \mathbb{R} \times (0,\infty), \\ u|_{t=0} = \varphi(x), &\quad x \in \mathbb{R}. \end{aligned}$$

(c)

$$\begin{aligned} u_t = a^2 u_{xx}, &\quad (x,t) \in (0,+\infty) \times (0,\infty), \\ u|_{t=0} = 0, &\quad x \in (0,+\infty), \\ u_x|_{x=0} = -1, &\quad t > 0. \end{aligned}$$

(3) Find the Green's function of half line problem

$$\begin{aligned} u_t - u_{xx} &= f, \quad x \in (0,+\infty), t > 0, \\ u|_{t=0} &= \varphi, \quad x \in (0,+\infty), \\ u_x|_{x=0} &= 0, \quad t > 0. \end{aligned}$$

And give the formal solution formula of this problem.

(4) Ω is a bounded open subset of $\mathbb{R}^n$, $Q = \Omega \times (0,T]$. If $u \in C^{2,1}(Q) \cap C(\bar{Q})$ is a solution of the following initial boundary value problem

$$\begin{aligned} u_t - \Delta u = f(x,t), &\quad (x,t) \in Q, \\ u|_{t=0} = \varphi(x), &\quad x \in \Omega, \\ u|_{\partial\Omega} = 1. & \end{aligned}$$

Try to prove there exists a constant C (depending on T and $|\Omega| = \int_\Omega dx$) such that the following inequality holds

$$\sup_{0\leq t\leq T} \int_\Omega u^2(x,t)dx + \int_0^T \int_\Omega |\nabla u|^2 dxdt \leq C\Big(\int_\Omega \varphi^2 dx + \int_0^T \int_\Omega f^2 dxdt + 1\Big).$$

(5) Find the formal solution of the following problem by using separation of variables

$$\begin{aligned} u_t &= a^2 u_{xx}, && (x,t) \in (0,1)\times(0,\infty), \\ u|_{t=0} &= x^2(1-x), && x \in (0,1), \\ u_x|_{x=0} &= u|_{x=1} = 0, && t > 0. \end{aligned}$$

(6) Ω is a bounded open subset of $\mathbb{R}^n$, $Q_T = \Omega \times [0,T)$. $c(x,t) \geq -c_0$ with some constant $c_0 > 0$. If $u \in C(\bar{Q}_T) \cap C^{2,1}(Q_T)$ satisfies

$$\begin{aligned} u_t - a^2 \Delta u + c(x,t)u \leq 0, &\quad (x,t) \in Q_T, \\ u|_{\partial_p Q_T} \leq 0. & \end{aligned}$$

Try to prove that $u \leq 0$ in Q_T. (Hint: try to use auxiliary function $e^{-c_0 t}u$)

4. Poisson's equation

4.1. **Fundamental solution.** The Poisson's equation in $\mathbb{R}^n$ reads

$$-\Delta u = f, \quad \text{in } \mathbb{R}^n. \tag{4.1}$$

We will first try to find some special solution formally. Since the Laplace operator is radially symmetric, it is natural to expect radially symmetric solutions. Assume $u(x) = v(|x|) = v(r)$, where $r = |x|$, then

$$u_{x_i} = v_r \frac{\partial r}{\partial x_i} = v_r \frac{x_i}{r}, \quad u_{x_i x_j} = v_{rr} \frac{x_i^2}{r^2} + v_r \left(\frac{1}{r} - \frac{x_i^2}{r^3}\right),$$

thus we can put this radially symmetric function into the homogeneous Poisson's equation $\Delta u = 0$, i.e.

$$\Delta u = v_{rr} + \frac{n-1}{r} v_r = 0, \quad \Rightarrow \quad (\log v_r)_r = \frac{1-n}{r}, \text{ in che case of } v_r \neq 0.$$

Consequently, there exist constants C and C' such that $v_r = Cr^{1-n}$ and

$$v(r) = \begin{cases} C \log r + C', & n = 2, \\ \dfrac{C}{r^{n-2}} + C', & n \geq 3. \end{cases}$$

Therefore, $u(x) = v(|x|)$ are solutions of Poisson's equation at any point but not at $x = 0$. Motivated by this formal computation, we give the following definition.

Definition 12. Let

$$\Phi(x) = \begin{cases} -\dfrac{1}{2\pi} \log |x|, & n = 2, \\ \dfrac{1}{n(n-2)\alpha(n)} \dfrac{1}{|x|^{n-2}}, & n \geq 3, \end{cases}$$

where $\alpha(n)$ is the volumn of n dimension ball. $\Phi(x)$ is called the **fundamental solution** of Poisson's equation.

Properties

(1) $|\nabla \Phi| \leq \dfrac{C}{|x|^{n-1}}$, $|D^2 \Phi| \leq \dfrac{C}{|x|^n}$ for $x \neq 0$.
(2) $\Delta \Phi = 0$ for $x \neq 0$ and $\Delta \Phi(x - y) = 0$ for $x \neq y$, $\forall y \in \mathbb{R}^n$.

Remark 4.1. The special choice of constants in the fundamental solution will be observed clearly in the following theorem.

Then we are able to represent the solution of Poisson's equation by using fundamental solution. More precisely we have the following theorem.

Theorem 4.1. *If $f \in C_0^2(\mathbb{R}^n)$, then $u = \Phi * f$ is a solution of problem* (4.1)

Proof. First we prove that $u \in C^2(\mathbb{R}^n)$. In fact,

$$\frac{u(x + he_i) - u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y) \frac{f(x + he_i - y) - f(x - y)}{h} dy.$$

Due to the fact that f has compact support, $\dfrac{\partial f(x-y)}{\partial x_i} = \lim_{h\to 0} \dfrac{f(x+he_i-y)-f(x-y)}{h}$ and Φ is locally integrable, by letting $h \to 0$, we have

$$\frac{\partial u}{\partial x_i} = \int_{\mathbb{R}^n} \Phi(y) \frac{\partial f}{\partial x_i}(x-y)dy.$$

By similar discussions, we can show that u is twice differentiable and

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \int_{\mathbb{R}^n} \Phi(y) \frac{\partial^2 f}{\partial x_i \partial x_j}(x-y)dy.$$

Next we will prove $-\Delta u = f$. $\forall \varepsilon > 0$ small enough,

$$\begin{aligned} -\Delta u(x) &= \int_{\mathbb{R}^n} \Phi(y)\Delta_x f(x-y)dy \\ &= \int_{B_\varepsilon(0)} \Phi(y)\Delta_x f(x-y)dy + \int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \Phi(y)\Delta_x f(x-y)dy \\ &=: I_\varepsilon + J_\varepsilon. \end{aligned}$$

where

$$|I_\varepsilon| \le C\|D^2 f\|_{L^\infty} \int_{B_\varepsilon(0)} |\Phi(y)|dy \le \left\{ \begin{array}{ll} C\varepsilon^2|\log\varepsilon| & n=2 \\ C\varepsilon^2 & n\ge 3 \end{array} \right. .$$

Integrating J_ε by parts, we have

$$J_\varepsilon = -\int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \nabla_y\Phi(y)\nabla_y f(x-y)dy - \int_{\partial B_\varepsilon(0)} \Phi(y)\nabla_y f(x-y)\cdot\gamma dS_y := K_\varepsilon + L_\varepsilon,$$

where L_ε can be estimated by

$$|L_\varepsilon| \le \|Df\|_{L^\infty} \int_{\partial B_\varepsilon(0)} |\Phi(y)|dS_y \le \left\{ \begin{array}{ll} C\varepsilon|\log\varepsilon|, & n=2, \\ C\varepsilon, & n\ge 3 \end{array} \right. .$$

In the end, we can see that K_ε contributes the main part of the calculation. When ε goes to 0, this term plays a role as a Delta function applied on f. Due to the fact that $\Delta\Phi(y) = 0$ for $y \neq 0$, we have

$$K_\varepsilon = \int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \Delta\Phi(y)f(x-y)dy + \int_{\partial B_\varepsilon(0)} \nabla\Phi\cdot\gamma f(x-y)dS_y = \int_{\partial B_\varepsilon(0)} \nabla\Phi\cdot\gamma f(x-y)dS_y.$$

Now we can calculate that on $\partial B_\varepsilon(0)$,

$$\nabla_y\Phi(y)\cdot\gamma = -\frac{1}{n\alpha(n)}\frac{y}{|y|^n}\frac{y}{|y|} = -\frac{1}{n\alpha(n)\varepsilon^{n-1}}.$$

Thus we have

$$K_\varepsilon = -\frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B_\varepsilon(0)} f(x-y)dS_y = -\frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B_\varepsilon(x)} f(y)dS_y.$$

In the end, taking $\varepsilon \to 0$ gives

$$K_\varepsilon \to f(x).$$

□

Remark 4.2. From the above proof, the delicate choosing of the constants that appear in the definition of the fundamental solution is now apparent and understandable.

By using the same method, we can prove that $-\Delta\Phi = \delta_0(x)$ in the sense of distribution.

Theorem 4.2.

$$\Phi(x,y) = \Phi(x-y) = \begin{cases} -\dfrac{1}{2\pi}\log|x-y|, & n=2 \\ \dfrac{1}{n(n-2)\alpha(n)}\dfrac{1}{|x-y|^{n-2}}, & n \geq 3 \end{cases} \tag{4.2}$$

is a solution of

$$-\Delta\Phi = \delta_y(x)$$

in the sense of distribution. More precisely, $\forall \varphi \in C_0^\infty(\mathbb{R}^n)$*, it holds*

$$\langle -\Delta\Phi(x-y), \varphi(x)\rangle = -\int_{\mathbb{R}^n} \Phi(x-y)\Delta\varphi(x)dx = \varphi(y) = \langle \delta_y(x), \varphi(x)\rangle = \varphi(y).$$

4.2. **Properties of harmonic functions.** Let Ω be an open subset of $\mathbb{R}^n$.

Definition 13. If $\Delta u = 0$ in Ω with $u \in C^2(\Omega)$, then u is called a **harmonic function** in Ω.

4.2.1. *Mean value theorem.*

Theorem 4.3. *If* $u \in C^2(\Omega)$ *is harmonic, then* $\forall$ *ball* $B(x,r) \in \Omega$*, it holds that*

$$u(x) = ⨍_{\partial B(x,r)} u dS_y = ⨍_{B(x,r)} u dy. \tag{4.3}$$

Proof. Let

$$w(r) = ⨍_{\partial B(x,r)} u(y) dS_y = ⨍_{\partial B(0,1)} u(x+rz) dS_z$$

Then by taking the derivative with respect to r, we have

$$\begin{aligned} w'(r) &= ⨍_{\partial B(0,1)} \nabla u(x+rz)\cdot z dS_z \\ &= ⨍_{\partial B(x,r)} \nabla u(y)\cdot \frac{y-x}{r} dS_y = \frac{r}{n|B(x,r)|}\int_{B(x,r)} \Delta u(y) dy = 0, \end{aligned}$$

which implies that $w(r)$ is a constant. Thus we have

$$w(r) = \lim_{s\to 0} w(s) = \lim_{s\to 0} ⨍_{\partial B(x,s)} u(y) dS_y = u(x).$$

For the mean value on $B(x,r)$, we know that

$$\begin{aligned} \int_{B(x,r)} u(y)dy &= \int_0^r \Big(\int_{\partial B(x,s)} u(y) dS_y\Big) ds \\ &= u(x)\int_0^r n\alpha(n)s^{n-1} ds = \alpha(n) r^n u(x), \end{aligned}$$

which is exactly

$$u(x) = \fint_{B(x,r)} u(y)dy.$$

□

Theorem 4.4. *If $u \in C^2(\Omega)$ such that*

$$u(x) = \fint_{\partial B(x,r)} u(y)dS_y, \quad \forall B(x,r) \subset \Omega,$$

then u is harmonic in Ω, i.e., $\Delta u = 0$ in Ω.

Proof. If $\Delta u \not\equiv 0$, there must exist a ball $B(x,r) \subset \Omega$ such that $\Delta u > 0$ in $B(x,r)$. On the other hand,

$$0 = w'(r) = \frac{r}{n} \fint_{B(x,r)} \Delta u(y)dy > 0,$$

which gives a contradiction. □

4.2.2. *Strong maximum principle.*

Theorem 4.5. *If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic in Ω, then*

(1) $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$
(2) *If Ω is connected and $\exists x_0 \in \Omega$ such that*

$$u(x_0) = \max_{\bar{\Omega}} u(x),$$

then u is constant within Ω.

Proof. The first statement is easy, we only prove that second one here.
Let

$$U_M = \{x \in \bar{\Omega} | u(x) = M = \max_{\bar{\Omega}} u(x)\} \subset \bar{\Omega}.$$

Then the set U_M is nonempty, furthermore it is a closed set because of the continuity of u. If $U_M \subset\subset \bar{\Omega}$, then $\exists x_1 \in \partial U_M \subset \Omega$ and $\forall 0 < r < dist(x_1, \partial\Omega)$ such that $u(x_1) = M$ and

$$M = u(x_1) = \fint_{B(x_1,r)} u(y)dy < M,$$

where the mean value property has been applied. We have arrived at a contradiction. □

Corollary 4.1. *If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic and $u \geq 0$ on $\partial\Omega$, then $u \geq 0$ in Ω.*

Corollary 4.2. *(Uniqueness) Dirichlet boundary value problem $-\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$ admits at most one $C^2(\Omega) \cap C(\bar{\Omega})$- solution.*

4.2.3. *Regularity.*

Theorem 4.6. *If $u \in C(\Omega)$ satisfies mean value property for all ball $B(x,r)$ in Ω, then $u \in C^\infty(\Omega)$.*

Remark 4.3. The smoothness up to $\partial\Omega$ is usually not true, which depends on the regularity of the boundary.

Proof. We use mollification to the proof of regularity. Let $j(x) \in C_0^\infty(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} j(x)dx = 1$. $\forall \varepsilon > 0$, we introduce the scaling

$$j_\varepsilon(x) = \frac{1}{\varepsilon^n} j(\frac{x}{\varepsilon}).$$

Let

$$\Omega_\varepsilon = \{x \in \Omega | dist(x, \partial\Omega) > \varepsilon\}.$$

Therefore $\forall x \in \Omega_\varepsilon$, $u_\varepsilon(x) = j_\varepsilon(x) * u(x) \in C^\infty(\Omega_\varepsilon)$, and with the help of the mean value property, we have

$$\begin{aligned} u_\varepsilon(x) &= \int_{B(x,\varepsilon)} \frac{1}{\varepsilon^n} j(\frac{x-y}{\varepsilon}) u(y)dy \\ &= \frac{1}{\varepsilon^n} \int_0^\varepsilon \left[j(\frac{r}{\varepsilon}) \int_{\partial B(x,r)} u(y) dS_y \right] dr \\ &= \frac{1}{\varepsilon^n} u(x) \int_0^\varepsilon j(\frac{r}{\varepsilon}) n\alpha(n) r^{n-1} dr \\ &= u(x) \int_{B(0,\varepsilon)} j_\varepsilon(y)dy = u(x). \end{aligned}$$

Thus $u(x) = u_\varepsilon(x) \in C^\infty(\Omega_\varepsilon)$, $\forall \varepsilon > 0$. □

4.2.4. *Liouville theorem.*

Theorem 4.7. *If $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded, then u is a constant.*

Proof. *** The proof will use local regularity estimates for harmonic function which was not talked about in this course. $\forall x_0 \in R^n$, $r > 0$,

$$|Du(x_0)| \leq \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x_0,r))} \leq \frac{C_1 \alpha(n)}{r} \|u\|_{L^\infty(\mathbb{R}^n)} \to 0, \text{ as } r \to \infty.$$

Then $Du \equiv 0$, which implies u is a constant. □

Corollary 4.3. *If $f \in C_0^2(\mathbb{R}^n)$, $n \geq 3$, then any bounded solution of $-\Delta u = f$ in $\mathbb{R}^n$ has the form*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy + C.$$

Proof. First we know that $\int_{\mathbb{R}^n} \Phi(x-y) f(y)dy$ is a bounded solution since $\Phi(x) \to 0$ as $|x| \to \infty$. If there is another bounded solution $\tilde{u}$, then $w = u - \tilde{u}$ is harmonic, thus by Liouville's theorem, w is a constant. □

4.3. **Green's Function.** The main goal of this subsection is to get the representation formula for the solution of boundary value problem

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= \varphi. \end{aligned} \tag{4.4}$$

Is the fundamental solution useful in getting a solution formula to the above Dirichlet problem?

Let's start from a formal calculation, $\forall x \in \Omega$,

$$\begin{aligned} u(x) &= \langle \delta_x(y), u(y) \rangle = \langle -\Delta_y \Phi(x,y), u(y) \rangle \\ &\sim -\int_\Omega \Delta_y \Phi(x,y) u(y) dy \\ &= \int_\Omega \Phi(x,y)(-\Delta_y u(y)) dy - \int_{\partial\Omega} \nabla_y \Phi(x,y) \cdot \gamma u(y) dS_y + \int_{\partial\Omega} \Phi(x,y) \nabla_y u(y) \cdot \gamma dS_y. \end{aligned}$$

Then formally, if $u|_{\partial\Omega} = \varphi$ and $-\Delta u = f$, we have

$$u(x) = \int_\Omega \Phi(x,y) f(y) dy - \int_{\partial\Omega} \nabla_y \Phi(x,y) \cdot \gamma \varphi(y) dS_y + \int_{\partial\Omega} \Phi(x,y) \nabla_y u(y) \cdot \gamma dS_y,$$

where however the last term is still unknown. We will try to consider another function $G(x,y)$ to replace the fundamental solution $\Phi(x,y)$. And this $G(x,y)$ satisfies, for any fixed $y \in \Omega$,

$$\begin{aligned} -\Delta_y G(x,y) &= \delta_x(y), \quad \forall y \in \Omega, \\ G(x,y)|_{y \in \partial\Omega} &= 0. \end{aligned}$$

A good candidate of $G(x,y)$ is $\Phi(x,y) + g(x,y)$ with $g(x,y)$ satisfing the following Dirichlet boundary value problem

$$\begin{aligned} -\Delta_y g(x,y) &= 0, \quad \forall x, y \in \Omega, \\ g|_{y \in \partial\Omega} &= -\Phi(x,y)|_{y \in \partial\Omega}. \end{aligned}$$

Once we can solve the above problem for $g(x,y)$, we will have the solution representation of (4.4),

$$u(x) = \int_\Omega G(x,y) f(y) dy - \int_{\partial\Omega} \nabla_y G(x,y) \cdot \gamma \varphi(y) dS_y.$$

We will give a proof of the above discussion after the following definition.

Definition 14. (Green's function)

$$G(x,y) = \Phi(x,y) + g(x,y)$$

is called the **Green's function** of (4.4), where $g(x,y) \in C^2(\Omega \times \Omega)$ is a solution of the following boundary value problem

$$\begin{aligned} -\Delta_y g(x,y) &= 0, \qquad \forall x, y \in \Omega, \\ g(x,y)|_{y \in \partial\Omega} &= -\Phi(x,y)|_{y \in \partial\Omega}. \end{aligned}$$

Theorem 4.8. *If Ω is an open subset of $\mathbb{R}^n$, $\partial\Omega$ is piecewise smooth and $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, then $\forall x \in \Omega$,*

$$u(x) = \int_\Omega \Phi(x,y)(-\Delta_y u(y))dy - \int_{\partial\Omega} \nabla_y \Phi(x,y) \cdot \gamma u(y) dS_y + \int_{\partial\Omega} \Phi(x,y) \nabla_y u(y) \cdot \gamma dS_y \tag{4.5}$$

Proof. $\forall \varepsilon > 0$ small enough, we have

$$\begin{aligned}
&\int_\Omega \Phi(x,y)(-\Delta_y u(y))dy = \lim_{\varepsilon\to 0^+} \int_{\Omega\setminus B_\varepsilon(x)} \Phi(x,y)(-\Delta_y u(y))dy \\
= &\lim_{\varepsilon\to 0^+} \int_{\Omega\setminus B_\varepsilon(x)} -\Delta_y \Phi(x,y) u(y) dy - \lim_{\varepsilon\to 0^+} \int_{\partial\Omega} (\Phi(x,y)\nabla u(y) \cdot \gamma - \nabla\Phi(x,y) \cdot \gamma u(y)) dS_y \\
&- \lim_{\varepsilon\to 0^+} \int_{\partial B(x,\varepsilon)} (\Phi(x,y)\nabla u(y) \cdot \gamma - \nabla\Phi(x,y) \cdot \gamma u(y)) dS_y \\
= &\ 0 - \lim_{\varepsilon\to 0^+} \int_{\partial\Omega} (\Phi(x,y)\nabla u(y) \cdot \gamma - \nabla\Phi(x,y) \cdot \gamma u(y)) dS_y + u(x),
\end{aligned}$$

where we have used the facts that

$$\left| \int_{\partial B(x,\varepsilon)} \Phi(x,y) \nabla u(y) \cdot \gamma dS_y \right| \le C\varepsilon \max_{\partial B(x,\varepsilon)} |\nabla u| \to 0, \text{ as } \varepsilon \to 0,$$

and

$$\int_{\partial B(x,\varepsilon)} u(y) \nabla\Phi(x,y) \cdot \gamma dS_y = ⨍_{\partial B(x,\varepsilon)} u(y) dS_y \to u(x), \text{ as } \varepsilon \to 0.$$

□

Theorem 4.9. *(Green's function is symmetric in its two variables)*

$$G(x,y) = G(y,x), \quad \forall x, y \in \Omega.$$

***Sketch of the proof. The technical point is the same as the proof of theorem 4.8. $\forall \varepsilon > 0$ small enough such that $B(x,\varepsilon) \cup B(y,\varepsilon) \subset \Omega$ and $B(x,\varepsilon) \cap B(y,\varepsilon) = \emptyset$, take $\Omega_\varepsilon = \Omega \setminus (B(x,\varepsilon) \cup B(y,\varepsilon))$. Notice that $G(x,z) = G(y,z) = 0$ on $z \in \partial\Omega$,

$$\begin{aligned}
0 &= \int_{\Omega_\varepsilon} (G(y,z)\Delta_z G(x,z) - G(x,z)\Delta_z G(y,z)) dz \\
&= \int_{\partial\Omega_\varepsilon} (G(y,z)\nabla_z G(x,z) \cdot \gamma - G(x,z)\nabla_z G(y,z) \cdot \gamma) dS_z \\
&= \int_{\partial B(x,\varepsilon)\cup\partial B(y,\varepsilon)} (G(y,z)\nabla_z G(x,z) \cdot \gamma - G(x,z)\nabla_z G(y,z) \cdot \gamma) dS_z.
\end{aligned}$$

We take $\partial B(y,\varepsilon)$ as an example,

$$\left| \int_{\partial B(y,\varepsilon)} G(y,z) \nabla_z G(x,z) \cdot \gamma dS_z \right| \le C(\varepsilon + \varepsilon^{n-1}) \to 0,$$

$$-\int_{\partial B(y,\varepsilon)} G(x,z) \nabla_z G(y,z) \cdot \gamma dS_z = ⨍_{\partial B(y,\varepsilon)} G(x,z) dS_z + o(\varepsilon^{n-1}) \to -G(x,y).$$

We can proceed the discussion in a similar manner for the integral on $\partial B(x,\varepsilon)$ and conclude that it converges to $G(y,x)$.

4.3.1. *Half space problem.* The half space we study here is $\mathbb{R}^n_+ = \{x = (x_1,\cdots,x_n) \in \mathbb{R}^n | x_n > 0\}$.

$\forall x = (x_1,\cdots,x_{n-1},x_n) \in \mathbb{R}^n_+$, we call $\tilde{x} = (x_1,\cdots,x_{n-1},-x_n)$ is x's reflection in the plane $\{x_n = 0\}$.

We study the following boundary value problem

$$\begin{aligned} -\Delta u = f, &\quad \text{in } \mathbb{R}^n_+, \\ u|_{\partial\mathbb{R}^n_+} = \varphi. & \end{aligned}$$

Our goal here is to find Green's function $G(x,y)$ of this problem and write the solution by using formula

$$u(x) = \int_\Omega G(x,y)f(y)dy - \int_{\partial\Omega} \nabla_y G(x,y)\cdot\gamma\varphi(y)dS_y.$$

$\forall x \in \mathbb{R}^n_+$, the Green's function should be a solution of

$$\begin{aligned} -\Delta_y G = \delta_x(y), &\quad y \in \mathbb{R}^n_+, \\ G|_{y\in\partial\mathbb{R}^n_+} = 0. & \end{aligned}$$

Then the Green's function of half space problem is easy to obtain, i.e.,

$$G(x,y) = \Phi(x,y) - \Phi(\tilde{x},y), \quad x,y \in \mathbb{R}^n_+, x \neq y,$$

and

$$\frac{\partial G}{\partial y_n}(x,y) = \frac{\partial \Phi}{\partial y_n}(y-x) - \frac{\partial \Phi}{\partial y_n}(y-\tilde{x}) = \frac{-1}{n\alpha(n)}\Big(\frac{y_n - x_n}{|y-x|^n} - \frac{y_n + x_n}{|y-\tilde{x}|^n}\Big).$$

Therefore, $\forall y \in \partial\mathbb{R}^n_+$,

$$\frac{\partial G}{\partial \gamma}(x,y) = -\frac{\partial G}{\partial y_n}(x,y) = -\frac{2x_n}{n\alpha(n)}\frac{1}{|x-y|^n}.$$

Then the solution of the boundary value problem in case $f \equiv 0$ can be represented by

$$u(x) = \frac{2x_n}{n\alpha(n)}\int_{\partial\mathbb{R}^n_+} \frac{\varphi(y)}{|x-y|^n}dy, \quad \forall x \in \mathbb{R}^n_+,$$

which is called the **Poisson's formula** of half space problem.

The function

$$K(x,y) := \frac{2x_n}{n\alpha(n)}\frac{1}{|x-y|^n}, \quad x \in R^n_+, y \in \partial\mathbb{R}^n_+$$

is called the **Poisson kernel** for $\mathbb{R}^n_+$.

Theorem 4.10. *Assume $\varphi \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$ and u is defined by the Poisson's formula. Then $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$, $-\Delta u = 0$ in $\mathbb{R}^n_+$ and $\forall x^0 \in \partial\mathbb{R}^n_+$,*

$$\lim_{x\in\mathbb{R}^n_+, x\to x^0} u(x) = \varphi(x^0).$$

Proof. The facts that $-\Delta u = 0$ and $\forall x \in \mathbb{R}^n_+$,

$$\int_{\partial\mathbb{R}^n_+} K(x,y)dy = 1,$$

are easy to check.

Since $\forall x \neq y$, $K(x,y)$ is a smooth function in x, we know directly that $u \in C^\infty(\mathbb{R}^n_+)$ and

$$\Delta u(x) = \int_{\partial\mathbb{R}^n_+} \Delta_x K(x,y)\varphi(y)dy = 0, \quad \forall x \in \mathbb{R}^n_+.$$

For boundary condition, $\forall x^0 \in \partial\mathbb{R}^n_+$, $\forall \varepsilon > 0$, choose $\delta > 0$ small enough such that $\forall y \in \partial\mathbb{R}^n_+$ and $|y - x^0| < \delta$, we have

$$|\varphi(y) - \varphi(x^0)| < \varepsilon.$$

Then $\forall x \in \mathbb{R}^n_+$ and $|x - x^0| < \delta/2$, we have

$$\begin{aligned}
&|u(x) - \varphi(x^0)| = \Big|\int_{\partial\mathbb{R}^n_+} K(x,y)(\varphi(y) - \varphi(x^0))dy\Big| \\
\leq\ & \int_{\partial\mathbb{R}^n_+ \cap B(x^0,\delta)} K(x,y)|\varphi(y) - \varphi(x^0)|dy + \int_{\partial\mathbb{R}^n_+ \setminus B(x^0,\delta)} K(x,y)|\varphi(y) - \varphi(x^0)|dy \\
\leq\ & \varepsilon + 2\|\varphi\|_{L^\infty} \int_{\partial\mathbb{R}^n_+ \setminus B(x^0,\delta)} K(x,y)dy \\
\leq\ & \varepsilon + \frac{2^{n+1}\|\varphi\|_{L^\infty} x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ \setminus B(x^0,\delta)} \frac{1}{|y - x^0|^n} dy \to 0, \qquad \text{as } x_n \to 0^+.
\end{aligned}$$

□

4.3.2. *Problem in a ball.* We will give an exact formula for the Green's function in a ball.

$\forall x \in B^n(0,1)$, the n dimensional unit ball. We need that $G(x,y) = 0$, $\forall y \in \partial B^n(0,1)$. Let $\tilde{x}$ be the inversion of x, i.e., $\tilde{x} = \dfrac{x}{|x|^2}$, thus

$$|\tilde{x} - y| \cdot |x| = |x - y|, \quad \forall y \in \partial B^n(0,1)$$

and in order that $G(x,y)$ satisfies the zero boundary condition, we set

$$G(x,y) = \Phi(|x-y|) - \Phi(|y-x|) = \Phi(|y-x|) - \Phi(|x|\cdot|y-\tilde{x}|), \quad \forall y \in \partial B^n(0,1).$$

Since Φ is the fundamental solution,

$$-\Delta_y \Phi(|x|\cdot|y-\tilde{x}|) = 0, \quad \forall y \neq \tilde{x}.$$

As a consequence,

$$G(x,y) = \Phi(|y-x|) - \Phi(|x|\cdot|y-\tilde{x}|), \quad \forall y \in B^n(0,1)$$

is called the Green's function for $B^n(0,1)$.

Now we will give the Poisson's formula for $B^n(0,1)$.

$$\begin{aligned}
-\Delta u = 0, &\quad \text{in } B^n(0,1), \\
u|_{\partial B(0,1)} = h.&
\end{aligned}$$

By Green's formula the solution is

$$u(x) = -\int_{\partial B(0,1)} h(y)\nabla G(x,y)\cdot\gamma dS_y.$$

We will explicitly calculate this formula.

$$\begin{aligned}
\nabla_y\Phi(y-x) &= -\frac{1}{n\alpha(n)}\frac{y-x}{|x-y|^n},\\
\nabla_y\Phi(|x|(y-\frac{x}{|x|^2})) &= -\frac{1}{n(n-2)\alpha(n)}\nabla_y\frac{1}{|x|^{n-2}|y-\frac{x}{|x|^2}|^{n-2}}\\
&= \frac{-1}{n\alpha(n)}\frac{1}{|x|^{n-2}}\frac{y-\frac{x}{|x|^2}}{|y-\frac{x}{|x|^2}|^n} = \frac{-1}{n\alpha(n)}\frac{y|x|^2-x}{[|x|(y-\frac{x}{|x|^2})]^n}\\
&= \frac{-1}{n\alpha(n)}\frac{y|x|^2-x}{|x-y|^n},
\end{aligned}$$

where we have used the fact that $y\in\partial B(0,1)$ and $|x|\cdot|y-\frac{x}{|x|^2}| = |x-y|$. Furthermore,

$$\begin{aligned}
\nabla_y G(x,y)\cdot\gamma|_{\partial B(0,1)} &= \frac{-1}{n\alpha(n)}\Big(\frac{y-x}{|x-y|^n}-\frac{y|x|^2-x}{|x-y|^n}\Big)\cdot y\Big|_{y\in\partial B(0,1)}\\
&= \frac{-1}{n\alpha(n)}\frac{|y|^2-x\cdot y-|y|^2|x|^2+x\cdot y}{|x-y|^n}\Big|_{|y|=1}\\
&= \frac{-1}{n\alpha(n)}\frac{|y|^2(1-|x|^2)}{|x-y|^n}\Big|_{|y|=1} = \frac{-1}{n\alpha(n)}\frac{1-|x|^2}{|x-y|^n}.
\end{aligned}$$

Thus the solution formula is

$$u(x) = \frac{1-|x|^2}{n\alpha(n)}\int_{\partial B(0,1)}\frac{h(y)}{|x-y|^n}dS_y.$$

For problems on $B(0,r)$, by doing scaling $\tilde{u}(x) = u(rx)$, $\tilde{h}(x) = h(rx)$, we will have the **Poisson's formula**

$$u(x) = \frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B(0,r)}\frac{h(y)}{|x-y|^n}dS_y, \quad \forall x\in B(0,r). \tag{4.6}$$

We call

$$\frac{r^2-|x|^2}{n\alpha(n)r}\frac{1}{|x-y|^n}$$

the **Poisson kernel** for $B(0,r)$.

Theorem 4.11. *If $h\in C(\partial B(0,r))$, then the function u given by (4.6) is in $C^\infty(B(0,r))$, $-\Delta u = 0$ and $\lim_{x\to x^0} u(x) = h(x^0)$, $\forall x^0\in\partial B$.*

4.4. **Maximum principle.** Here we consider more general equations. Let Ω be a bounded open subset of $\mathbb{R}^n$ and

$$Lu = -\Delta u + c(x)u, \quad \text{in } \Omega.$$

Theorem 4.12. *(Weak maximum principle) Let $0 \le c(x) \le \bar{c}$ in Ω, if $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and $Lu \le 0$ in Ω, then*

$$\max_{\bar{\Omega}} u(x) \le \max_{\partial\Omega} u^+(x),$$

where $u^+(x) = \max\{u(x), 0\}$.

Proof. Assume $Lu < 0$ in Ω. If $\exists x_0 \in \Omega$ such that

$$0 \le u(x_0) = \max_{\bar{\Omega}} u,$$

then

$$-\Delta u|_{x=x_0} + c(x_0)u(x_0) \ge 0,$$

which is a contradiction.

If $Lu \le 0$ in Ω, we introduce an auxiliary function

$$w(x) = u(x) + \varepsilon e^{ax_1},$$

where a is chosen such that $-a^2 + \bar{c} < 0$. Therefore

$$Lw = Lu + \varepsilon e^{ax_1}(-a^2 + c(x)) < 0.$$

By using the above discussion, we have $\max\limits_{\bar{\Omega}} w \le \max\limits_{\partial\Omega} w^+$, then the weak maximum principle follows by taking $\varepsilon \to 0$.

□

Remark 4.4. If $c \equiv 0$, then $\max\limits_{\partial\Omega} u^+$ in the theorem can be replaced by $\max\limits_{\partial\Omega} u$.

Remark 4.5. If $Lu \ge 0$ in Ω, then $\min\limits_{\bar{\Omega}} u \ge \min\limits_{\partial\Omega}(-u^-)$ where $u^-(x) = \max\{-u(x), 0\}$.

Next, by using the above weak maximum principle, we will get a maximum estimate for the following boundary value problem

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega, \\ u &= \varphi, \quad \text{on } \partial\Omega. \end{aligned} \tag{4.7}$$

Theorem 4.13. *If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a solution of (4.7), then*

$$\max_{\bar{\Omega}} |u| \le \Phi + CF,$$

where $\Phi = \max\limits_{\partial\Omega} |\varphi|$, $F = \sup\limits_{\bar{\Omega}} |f|$, C is a constant depending on n and $\operatorname{diam}\Omega$.

Proof. Without loss of generality, let $x = 0 \in \Omega$ and

$$w(x) = \pm u + \frac{F}{2n}(d^2 - |x|^2) + \Phi,$$

then

$$-\Delta w = \pm f + F \geq 0, \quad w|_{\partial\Omega} \geq \Phi \pm \varphi \geq 0.$$

By comparison principle, we have $w \geq 0$ in $\bar{\Omega}$, which implies

$$\max_{\Omega} |u| \leq \Phi + \frac{F}{2n} d^2.$$

□

4.5. **Variational problem.** We show in this part that the solvability of the boundary value problem of Poisson's equation is equivalent to the solvability of a variational problem. Namely

$$\begin{aligned} -\Delta u &= f, \quad \text{ in } \Omega, \\ u &= g, \quad \text{ on } \partial\Omega \end{aligned} \tag{4.8}$$

is equivalent to the following problem

$$\begin{aligned} J(u) &= \inf_{v \in M_g} J(v), \\ J(v) &= \frac{1}{2}\int_{\Omega} |\nabla v|^2 dx - \int_{\Omega} f v dx, \\ M_g &= \{v \in C^1(\bar{\Omega}) | v = g \text{ on } \partial\Omega\}. \end{aligned} \tag{4.9}$$

4.5.1. *Dirichlet principle.*

Theorem 4.14. *(Dirichlet principle) Assume $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, then u is a solution of* (4.8) *if and only if u is a solution of* (4.9)*.*

Proof. "⇒". $\forall v \in M_g$, we choose $u - v$ as test function in (4.8),

$$\int_{\Omega} -\Delta u (u - v) = \int_{\Omega} f(u - v).$$

Integration by parts with boundary condition $u - v = 0$ on $\partial\Omega$ shows

$$\int_{\Omega} \nabla u \cdot \nabla (u - v) = \int_{\Omega} f(u - v).$$

Equivalently,

$$\int_{\Omega} |\nabla u|^2 - \int_{\Omega} f u = \int_{\Omega} \nabla u \cdot \nabla v - \int_{\Omega} f v \leq \frac{1}{2}\int_{\Omega} |\nabla u|^2 + \frac{1}{2}\int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v.$$

Then we have

$$\frac{1}{2}\int_{\Omega} |\nabla u|^2 - \int_{\Omega} f u \leq \frac{1}{2}\int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v,$$

which is

$$J(u) \leq J(v), \quad \forall v \in M_g.$$

"⇐" $\forall v \in M_0$, we have $u + \varepsilon v \in M_g$. Let $j(\varepsilon) = J(u + \varepsilon v)$, since u is a solution of (4.9), we know that $j'(\varepsilon)|_{\varepsilon=0} = 0$. More precisely,

$$\begin{aligned} &\frac{d}{d\varepsilon}\left[\int_\Omega \frac{1}{2}|\nabla(u+\varepsilon v)|^2 - \int_\Omega f(u+\varepsilon v)\right]\Big|_{\varepsilon=0} \\ = &\int_\Omega \nabla(u+\varepsilon v)|_{\varepsilon=0}\cdot\nabla v - \int_\Omega fv = \int_\Omega \nabla u\cdot\nabla v - \int_\Omega fv = \int_\Omega(-\Delta u - f)v, \end{aligned}$$

which holds for any $v \in C_0^1(\Omega)$. Thus u is a solution of (4.9).

□

The equation $-\Delta u = f$ in Ω is called the **Euler-Lagrange equation** of the variational problem (4.9).

In the 19th century, it is thought that variational problem always has a solution in the given set of functions. But Weierstrass gave an example which shows that the infimum can not be achieved by a function in the given set, i.e., the minimizer doesn't exist. Here is the example,

Example 12. *** (Weierstrass) Variational problem. Let $M = \{\varphi(x) \in C[0,1] | \varphi'(x)$ is continuous except finite removable or jump discontinuous points and $\varphi(0) = 1, \varphi(1) = 0\}$. The functional is

$$J(\varphi) = \int_0^1 [1 + (\varphi')^2]^{\frac{1}{4}} dx.$$

It is obvious that $\inf_{\varphi\in M} J(\varphi) = 1$. In fact, we only need to prove $\forall \delta > 0$, $\exists \varphi_\delta \in M$ such that

$$J(\varphi_\delta) \le 1 + \delta,$$

where we can choose

$$\varphi_\delta = \begin{cases} \frac{1}{\delta^2}(\delta^2 - x), & 0 \le x \le \delta^2 \\ 0, & \delta^2 < x \le 1 \end{cases}.$$

On the other hand, we couldn't find any $\varphi \in M$ such that $J(\varphi) = 1$. Otherwise, $\varphi' = 0$ a.e., then $\varphi \equiv C$, which contradicts with $\varphi(0) = 1$, $\varphi(1) = 0$.

Another fact is that even if the boundary value problem (4.8) has a solution in $C^2(\Omega) \cap C(\bar\Omega)$, it may not be obtained by solving the variational problem (4.9). Here is an example by Hadamard.

Example 13. *** $\Omega = B(0,1)$, $f \equiv 0$, $\varphi(\theta) = \displaystyle\sum_{n=1}^{\infty} \frac{\sin n^4\theta}{n^2} \in C(\partial\Omega)$, $0 \le \theta \le 2\pi$.

We know that (4.8) has a unique solution $u_0 \in C(\bar\Omega) \cap C^2(\Omega)$ with expression

$$u_0(\rho,\theta) = \sum_{n=1}^{\infty} \frac{\sin n^4\theta}{n^2}\rho^{n^4}.$$

On the other hand we can prove that

$$J(u_0) = +\infty.$$

In fact,

$$\begin{aligned} J(u_0) &= \lim_{r\to 1-} \int\int_{\rho\le r} |\nabla u_0|^2 dxdy = \lim_{r\to 1-} \int\int_{\rho\le r} \Big[\Big(\frac{\partial u_0}{\partial \rho}\Big)^2 + \frac{1}{\rho^2}\Big(\frac{\partial u_0}{\partial \theta}\Big)^2\Big]\rho d\rho d\theta \\ &= \lim_{r\to 1-} 2\pi \int_0^r \sum_{n=1}^{\infty} n^4 \rho^{2n^4-1} d\rho = \lim_{r\to 1-} \pi \sum_{n=1}^{\infty} r^{2n^4} = +\infty. \end{aligned}$$

4.6. **Sobolev space $H^1(\Omega)$ and $H_0^1(\Omega)$.** The Sobolev space $H^1(\Omega)$ is defined by

$$H^1(\Omega) = \{u \in \mathcal{D}'(\Omega) | u, Du \in L^2(\Omega)\},$$

where the inner product and the norm are given respectively by

$$\langle u, v\rangle_{H^1} = \int_\Omega uv + \int_\Omega \nabla u \cdot \nabla v, \quad \|u\|_{H^1} = \|u\|_{L^2} + \|\nabla u\|_{L^2}.$$

H^1 is a Hilbert space.

$H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with H^1-norm, so it is a subspace of $H^1(\Omega)$.

Remark 4.6. The Poincaré's inequality implies that $\|u\|_{H^1}$ and $\|\nabla u\|_{L^2}$ are equivalent norms in H^1 space.

*****The most important property for Sobolev spaces is the compact embedding theorem. For bounded Ω with uniform cone condition, $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$. Therefore, $(-\Delta)^{-1}$ with homogenous Dirichlet boundary condition is a compact operator in $L^2(\Omega)$, since

$$(-\Delta)^{-1} : L^2(\Omega) \to H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega).$$

Definition 15. If $\exists u \in H_0^1(\Omega)$ such that

$$J(u) = \inf_{v\in H_0^1} \Big(\frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega fv\Big)$$

we call u is a solution of (4.9).

Definition 16. If $\exists u \in H_0^1$ such that $\forall v \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv,$$

then we call u a weak solution of (4.8).

Theorem 4.15. *If $u \in H_0^1(\Omega)$, then u is a weak solution of* (4.8) *if and only if u is a solution of* (4.9)*.*

The proof of this theorem is left to readers.

4.7. **Solvability of variational problem.** Our goal in this subsection is to prove the unique solvability of variational problem (4.9).

Theorem 4.16. *Solution of* (4.9) *in* $H_0^1(\Omega)$ *is unique.*

Proof. Let $u_1, u_2 \in H_0^1(\Omega)$ be two solutions of (4.9), i.e.,

$$J(u_1) = J(u_2) = m = \inf_{v \in H_0^1(\Omega)} J(v),$$

then

$$0 = \frac{1}{2}\int_\Omega |\nabla u_1|^2 - \frac{1}{2}\int_\Omega |\nabla u_2|^2 - \int_\Omega (u_1 - u_2) f.$$

Noticing the fact that

$$\left|\frac{\nabla(u_1 - u_2)}{2}\right|^2 + \left|\frac{\nabla(u_1 + u_2)}{2}\right|^2 = \frac{1}{2}|\nabla u_1|^2 + \frac{1}{2}|\nabla u_2|^2,$$

we have

$$\begin{aligned}
\int_\Omega \left|\frac{\nabla(u_1 - u_2)}{2}\right|^2 &= \int_\Omega \frac{1}{2}|\nabla u_1|^2 + \int_\Omega \frac{1}{2}|\nabla u_2|^2 - \int_\Omega \left|\frac{\nabla(u_1 + u_2)}{2}\right|^2 \\
&\quad - \int_\Omega u_1 f - \int_\Omega u_2 f + 2\int_\Omega \frac{u_1 + u_2}{2} f \\
&= J(u_1) + J(u_2) - 2J(\frac{u_1 + u_2}{2}) \le 0
\end{aligned}$$

which implies that

$$\|\nabla(u_1 - u_2)\|_{L^2} = 0.$$

Poincaré's inequality gives

$$\|u_1 - u_2\|_{L^2} = 0 \quad \Rightarrow \quad u_1 = u_2 \text{ a.e. in } \Omega.$$

$\square$

Theorem 4.17. *(Existence)* $f \in L^2(\Omega)$, *then* (4.9) *has a solution* $u \in H_0^1(\Omega)$.

Proof. First we prove that $J(u)$ has a lower bound. In fact, by Young's and Poincaré's inequality,

$$J(v) = \frac{1}{2}\|\nabla v\|_{L^2}^2 - \int_\Omega fv \ge \frac{1}{2}\|\nabla v\|_{L^2}^2 - \frac{1}{4}\|\nabla v\|_{L^2}^2 - C\|f\|_{L^2}^2 \ge -C(d)\|f\|_{L^2}^2.$$

Let

$$m = \inf_{v \in H_0^1(\Omega)} J(v).$$

Let $\{v_k\}_{k=1}^\infty \subset H_0^1(\Omega)$ be a minimizing sequence such that

$$J(v_k) \le m + \frac{1}{k}.$$

Actually, $\{v_k\}$ is a Cauchy sequence in $H^1(\Omega)$ because of the following inequality which comes from similar discussions to the proof of uniqueness, for $k, l \to \infty$,

$$\left\|\nabla \frac{(v_k - v_l)}{2}\right\|_{L^2}^2 = J(v_k) + J(v_l) - 2J(\frac{v_k + v_l}{2}) \le m + \frac{1}{k} + m + \frac{1}{l} - 2m \le \frac{1}{k} + \frac{1}{l} \to 0.$$

Due to the completeness of H_0^1, there must $\exists u \in H_0^1(\Omega)$ such that

$$v_k \to u, \quad \text{in } H^1(\Omega).$$

Taking limit as $k \to \infty$, we have $J(v_k) \to J(u)$ and $J(u) = m$.

□

4.8. *****Lax-Milgram theorem and existence.** We first list the Lax-Milgram theorem from functional analysis, then prove the existence of weak solution of (4.8).

Theorem 4.18 (Lax-Milgram theorem)**.** *Let H be a Hilbert space. Assume $a(u,v)$ is a bi-linear mapping from $H \times H$ to $\mathbb{R}$ and satisfies*

- *Bounded. $\exists M \ge 0$ such that $|a(u,v)| \le M\|u\| \cdot \|v\|$, $\forall u, v \in H$.*
- *Coercive. $\exists \delta > 0$ such that $a(u,u) \ge \delta \|u\|^2$, $\forall u \in H$.*

Then for any bounded linear functional $F(v)$ on H, there exists a unique $u \in H$ such that

$$F(v) = a(u,v), \quad \forall v \in H.$$

and

$$\|u\| \le \frac{1}{\delta}\|F\|.$$

Proof. For any fixed $u \in H$, Riesz representation theorem implies that $\exists Au \in H$ such that

$$a(u,v) = (Au, v), \quad \forall v \in H.$$

The linearity of Au in u is obvious due to the fact that $a(u,v)$ is linear in u. Furthermore,

$$(Au, v) \le M\|u\| \cdot \|v\|, \quad \Rightarrow \quad \|Au\| \le M\|u\|.$$

Coercivity gives that $\forall u \in H$,

$$\delta\|u\|^2 \le a(u,u) = (Au, u) \le \|Au\| \cdot \|u\|, \quad \Rightarrow \quad \|Au\| \ge \delta\|u\|.$$

Thus A^{-1} exists. We claim that $R(A) = H$.

First $R(A)$ is closed. In fact, choose any Cauchy sequence $\{Au_k\}$ in $R(A)$, then $\lim_{k\to\infty} Au_k = v$. By coercivity, we have

$$\delta\|u_k - u_l\| \le \|Au_k - Au_l\|,$$

which means $\{u_k\}$ is also a Cauchy sequence in H. $\exists u \in H$ such that

$$\lim_{k\to\infty} u_k = u.$$

Thus

$$Au = \lim_{k\to\infty} Au_k = v.$$

If $R(A) \neq H$, $\exists w \neq 0$ in H such that

$$(Au, w) = 0, \quad \forall u \in H,$$

which contradicts with coercivity if we choose $w = u$. Thus $R(A) = H$.

For any linear functional $F(v)$ on H, by Riesz representation theorem, we have a unique $w \in H$ s.t.

$$F(v) = (w, v).$$

Let $u = A^{-1}w$, we have

$$\|u\| \leq \|A^{-1}\| \cdot \|w\| \leq \frac{1}{\delta}\|F\|$$

and

$$F(v) = (Au, v).$$

□

Theorem 4.19. *For $f \in L^2(\Omega)$, there exists a solution $u \in H_0^1(\Omega)$ of (4.8).*

Proof. Let the bilinear functional be defined by

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v.$$

Then it is coercive

$$a(u, u) \geq \|\nabla u\|_{L^2}^2 \geq C\|u\|_{H^1}^2.$$

Lax-Milgram theorem implies that $\forall f \in L^2(\Omega)$, there exists a unique $u \in H_0^1(\Omega)$ such that

$$a(u, v) = \langle f, v\rangle, \quad \forall v \in H_0^1(\Omega).$$

□

4.9. **Energy Estimate.** Energy methods for Poisson's equation is easy. I will not talk about it here, but leave it as an exercise. The energy estimate also shows that $-\Delta u = f$ in Ω and $u = h$ on $\partial\Omega$ has at most one solution in $C^2(\Omega) \cap C^1(\bar{\Omega})$.

4.10. **Problems.**

(1) Try to derive energy estimates for Dirichlet problem of Possion equation.
(2) Modify the proof of the mean value formulas to show for $n \geq 3$ that

$$u(0) = \fint_{\partial B(0,r)} g dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right) f dx,$$

provided $\begin{cases} -\Delta u = f, & x \in B(0,r) \\ u = g, & x \in \partial B(0,r) \end{cases}.$

(3) We say $v \in C^2(\bar{\Omega})$ is subharmonic if $-\Delta v \leq 0$ in Ω.
   (a) Prove for subharmonic v that

$$v(x) \leq \fint_{B(x,r)} v dy, \quad \text{for all } B(x,r) \subset \Omega.$$

   (b) Prove that therefore $\max_{\bar{\Omega}} v = \max_{\partial\Omega} v$.
   (c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.
   (d) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic.

(4) Let $B^+(R) = \{(x,y) : x^2 + y^2 < R^2, y > 0\}$, try to find the Green's function of the following problem

$$\begin{cases} -\Delta u = f(x,y), & (x,y) \in B^+(R), \\ u|_{\partial B^+(R)\cap\{y>0\}} = \varphi(x,y), & \\ u_y|_{y=0} = \psi(x,0), & -R \le x \le R. \end{cases}$$

Furthermore, give the representation formula of solution.

(5) Ω is a bounded open subset of $\mathbb{R}^n$, $u(x)$ is a classical solution of

$$\begin{cases} -\Delta u + c(x)u = f(x), & x \in \Omega, \\ (\nabla u \cdot \gamma + \alpha(x)u)|_{\Gamma_1} = \varphi_1, & u|_{\Gamma_2} = \varphi_2 \end{cases}$$

where $\Gamma_1 \cup \Gamma_2 = \partial\Omega$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_2 \neq \emptyset$.

If $c(x) \ge 0$, $\alpha(x) \ge \alpha_0 > 0$, try to prove the following estimate,

$$\max_{\Omega} |u(x)| \le C(\alpha_0, \mathrm{diam}\Omega)\Big[\sup_{\Omega} |f| + \sup_{\Gamma_1} |\varphi_1| + \sup_{\Gamma_2} |\varphi_2|\Big].$$

(6) Try to get the Euler-Lagrange equation of the following variational problem

$$J(u) = \min_{v \in M_0} J(v), \text{ with } M_0 = \{u \in C^2(\Omega) \cap C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0\},$$

(a) $J(v) = \displaystyle\int_\Omega (\frac{1}{p}|\nabla v|^p - fv)dx$, $p > 1$

(b) $J(v) = \displaystyle\int_\Omega (\frac{1}{2m}|\nabla v^m|^2 - fv)dx$, $m > 0$

(c) $j(v) = \displaystyle\int_\Omega (\sqrt{1 + |\nabla v|^2}dx + v^p)dx$, $p > 1$

(7) If $u \in H_0^1(\Omega)$ is a weak solution of

$$-\Delta u + u = f,$$

prove that u is a solution of variational problem

$$J(u) = \min_{v \in H_0^1(\Omega)} J(v),$$

where $J(v) = \dfrac{1}{2}\int_\Omega |\nabla v|^2 dx + \dfrac{1}{2}\int_\Omega v^2 dx - \int_\Omega fv dx$.

(8) Assume $f \in L^2(\Omega)$, $\varphi \in H^1(\Omega)$, $c(x) \ge 0$ and $c(x) \in C(\bar{\Omega})$, prove that variational problem

$$J(u) = \min_{v \in M_\varphi} J(v)$$

has a unique solution in $M_\varphi = \{u \in H^1(\Omega) : u - \varphi \in H_0^1(\Omega)\}$, where

$$J(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 + c(x)v^2 - fv)dx.$$

Furthermore, show that the solution of variational problem is a weak solution of

$$-\Delta u + c(x)u = f \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega.$$

5. Conservation Laws

Let $u(x,t)$ be the density and $q(u(x,t))$ be the flux function of a flow. In 1-dimensional case, for any interval $(x_1, x_2) \in \mathbb{R}$, we know that the change of total mass in (x_1, x_2) with respect to t equals to the difference between the flux function at the endpoint, i.e.,

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)dx = -q(u(x_2,t)) + q(u(x_1,t)).$$

If furthermore u and q are smooth functions, we have

$$\int_{x_1}^{x_2} \Big(u_t + [q(u(x,t))]_x\Big)dx = 0.$$

Now because of the arbitrariness of (x_1, x_2), we have the one dimensional conservation law in the differential form,

$$u_t + [q(u)]_x = 0, \qquad \text{in } \mathbb{R}, \tag{5.1}$$

- If $q(u) = vu$, with v being a given function, then the equation is a linear transport equation which was introduced in the beginning of this course.
- If $q(u) = -\kappa\nabla u$, then the equation is reduced into the heat equation. This relation is from Fourier's law in physics, i.e., the local heat flux density $q(u)$ is equal to the product of thermal conductivity κ and the negative local temperature gradient $-\nabla u$, where the heat flux density is the amount of energy that flows through a unit area per unit time.
- If $q(u)$ is a given nonlinear function of u, then the equation is called nonlinear conservation law. We will give two examples in this part: Burger's equation and the traffic flow equation.
  - Burger's equation. $q(u) = \frac{1}{2}u^2$.
  - Traffic flow problem. $q(u) = v(u)u$, with $v(u) = v_m(1 - \frac{u}{u_m})$, where v_m is the maximum speed of the cars and u_m is the maximum density.

5.1. **Local existence and smooth solutions.** We will take Burger's equation as an example to show the local existence, and similar results hold also for traffic flow problem.

$$\begin{aligned} &\rho_t + \rho\rho_x = 0, \quad x \in \mathbb{R}, t > 0, \\ &\rho|_{t=0} = g(x), \quad x \in \mathbb{R}. \end{aligned} \tag{5.2}$$

Formally, by using characteristic method, the Cauchy problem of Burger's equation is reduced into

$$\frac{d}{dt}\rho(x(t,x_0),t)) = 0, \quad \rho(x(0),0) = g(x_0),$$

where the characteristic lines $x(t,x_0)$ satisfies

$$\frac{dx}{dt} = \rho(x,t), \quad x|_{t=0} = x_0. \quad \Leftrightarrow \quad x = x_0 + \rho(x,t)t$$

Then the solution of Burger's equation satisfies an equation

$$\rho(x,t) = g(x - \rho(x,t)t).$$

Theorem 5.1. *If the initial data $g(x) \in C^1(\mathbb{R})$ and*

$$\min_{x\in\mathbb{R}} g'(x) \geq -a > -\infty, \text{ for some } a \geq 0,$$

then problem (5.2) has a unique C^1-solution in the domain $\{(x,t) : x \in \mathbb{R}, 0 \leq t < 1/a\}$.

Proof. For any given small $\varepsilon > 0$, let $F(x,t,\rho) = \rho - g(x - \rho t)$, the partial derivative of F in ρ can be estimated in the following inequality, $\forall (x,t) \in \bar{Q}_{a\varepsilon} = \{(x,t) : x \in \mathbb{R}, 0 \leq t \leq 1/a - \varepsilon\}$,

$$F_\rho(x,t,\rho) = 1 + g'(x - \rho t)t \geq 1 - at \geq a\varepsilon > 0.$$

With the help of implicit function theorem, we conclude that

$$F(x,t,\rho) = 0$$

has a unique solution $\rho \in C^1(\bar{Q}_{a\varepsilon})$. Therefore the unique solvability of the problem is obtained by arbitrariness of ε. $\square$

For general equations, the Cauchy problem is

$$\begin{aligned} &u_t + [q(u)]_x = 0, \quad x \in \mathbb{R}, t > 0, \\ &u|_{t=0} = g(x), \quad x \in \mathbb{R}. \end{aligned} \tag{5.3}$$

The smooth solution exists globally in the case that the second derivative of q and the first derivative of g have the same sign, which means that the characteristics don't interact each other.

Theorem 5.2. *If $q \in C^2$, $g \in C^1$ and either $q'' \geq 0$ and $g' \geq 0$ or $q'' \leq 0$ and $g' \leq 0$, then the C^1 solution is uniquely determined by*

$$u(x,t) = g(x - q'(u)t). \tag{5.4}$$

With the method of characteristics we know that the first discontinuity appears at time t such that

$$1 + tq''(u)g'(x - q'(u)t) = 0.$$

For example, in the case $q(u) = u - u^2$ and $g(x) = \arctan x$,

$$q'(u) = 1 - 2u, \quad q''(u) = -2 < 0, \quad g(x) = \arctan x, \quad g'(x) = \frac{1}{1+x^2} > 0.$$

For any $(x,t) \in \mathbb{R} \times (0, +\infty)$, let x_0 be the initial point that can be connected to (x,t) by characteristics. Since $u(x,t)$ is constant along characteristics, i.e. $u(x,t) = g(x_0)$, we have

$$q''(u(x,t))g'(x - q'(u(x,t))t) = q''(g(x_0))g'(x_0) = \frac{-2}{1+x_0^2}.$$

The first time that the continuity of the solution breaks down appears at time t such that

$$1 + t\Big(\frac{-2}{1+x_0^2}\Big) = 0.$$

This means the possible time is $t_s = \frac{1}{2}$ at point $x_0 = 0$.

At time $0 \leq t < \frac{1}{2}$, we know the solution is uniquely determined by

$$u - \arctan(x - (1-2u)t) = 0.$$

At time $t > \frac{1}{2}$, it would appear multi-valued functions near $t_s = \frac{1}{2}$.

This example also tells us that we need more theory on conservation law with discontinuous initial data. In this course, we only introduce the Cauchy problem with simplest discontinuous initial data, which is called Riemann problem. We will explain the theory of Riemann problem for traffic flow equation in the next section. The same theory holds for Burger's equation.

5.2. **Riemann problem for traffic flow and Burger's equation.** We will explain the theory of Riemann problem for traffic flow equation in the first subsection in detail, and leave the corresponding detailed analysis for Burger's equation to the readers.

5.2.1. *Traffic flow problem.* we will use ρ to represent the density of cars. The Cauchy problem is

$$\begin{aligned} \rho_t + v_m(1 - \frac{2\rho}{\rho_m})\rho_x = 0, \qquad & \text{in } \mathbb{R} \times (0, +\infty), \\ \rho|_{t=0} = g(x), \qquad & \end{aligned} \tag{5.5}$$

where ρ_m is the maximum density and v_m is the maximum velocity.

By the method of characteristics, formally we know that the density is conserved along the characteristics. We can rewrite (5.5) into the following equivalent problems

$$\begin{aligned} \frac{d}{dt}\rho(x(t,x_0),t) = 0, \qquad & \rho(x(0,x_0),0) = g(x_0), \\ \dot{x}(t) = v_m\Big(1 - \frac{2\rho}{\rho_m}\Big), \qquad & x|_{t=0} = x_0, \end{aligned}$$

which means $\rho(x(t),t) = g(x_0)$ and

$$\begin{aligned} & \dot{x}(t) = v_m\Big(1 - \frac{2g(x_0)}{\rho_m}\Big), \\ \Rightarrow \quad & x(t) = v_m\Big(1 - \frac{2g(x_0)}{\rho_m}\Big)t + x_0, \\ \Rightarrow \quad & x_0 = x - v_m\Big(1 - \frac{2g(x_0)}{\rho_m}\Big)t. \end{aligned}$$

Thus the formal solution of traffic flow problem is

$$\rho(x,t) = g\Big(x - v_m\Big(1 - \frac{2g(x_0)}{\rho_m}\Big)t\Big), \tag{5.6}$$

where $v_m\Big(1 - \dfrac{2g(x_0)}{\rho_m}\Big) = q'(g(x_0))$ is the traveling wave propagation speed.

From here one can see that the existence of classical solution totally depends on the shape of initial data $g(x)$.

We will study the two typical initial data.

Green light problem Initial data

$$g(x) = \begin{cases} \rho_m, & x \le 0, \\ 0, & x > 0. \end{cases} \tag{5.7}$$

In this case, the wave speed is

$$q'(g(x)) = \begin{cases} -v_m, & x \le 0, \\ v_m, & x > 0. \end{cases} \tag{5.8}$$

It means that in (x,t) plan, the solution can be determined in the domains on the right hand side of $x > v_m t$ and on the left hand side of $x < -v_m t$, i.e.,

$$\rho(x,t) = \begin{cases} \rho_m, & x < -v_m t, \\ ?, & -v_m t \le x \le v_m t, \\ 0, & x > v_m t. \end{cases}$$

The above analysis shows that by the method of characteristics, we don't know how to determine the value of solution in the domain $-v_m t < x < v_m t$. One way to find the reasonable representation of solution inside of this domain is to use an approximation of the initial data. More precisely, we will use function $g_\varepsilon(x)$ instead of the initial data $g(x)$,

$$g_\varepsilon(x) = \begin{cases} \rho_m, & x \le 0, \\ \rho_m(1 - \frac{x}{\varepsilon}), & 0 < x < \varepsilon, \\ 0, & x > \varepsilon. \end{cases} \tag{5.9}$$

It is easy to calculate the characteristics of this problem, i.e.,

$$x = \begin{cases} -v_m t + x_0, & x_0 < 0, \\ -v_m(1 - 2\frac{x_0}{\varepsilon})t + x_0, & 0 \le x_0 < \varepsilon, \\ v_m t + x_0, & x_0 \ge \varepsilon. \end{cases}$$

The characteristics inside of the region $0 \le x_0 < \varepsilon$ looks like a rarefaction fan.

Now the solution of traffic flow problem with initial data (5.9) (by using characteristic method) is

$$\rho_\varepsilon(x,t) = \begin{cases} \rho_m, & x < -v_m t, \\ \rho_m(1 - \frac{x + v_m t}{2v_m t + \varepsilon}), & -v_m t < x < v_m t + \varepsilon, \\ 0, & x > v_m t + \varepsilon. \end{cases} \tag{5.10}$$

Letting $\varepsilon \to 0$ yields

$$\rho(x,t) = \begin{cases} \rho_m, & x < -v_m t, \\ \frac{\rho_m}{2}(1 - \frac{x}{v_m t}), & -v_m t < x < v_m t, \\ 0, & x > v_m t. \end{cases} \tag{5.11}$$

We can see that the solution in the "fan" is a self-similar solution of the equation.

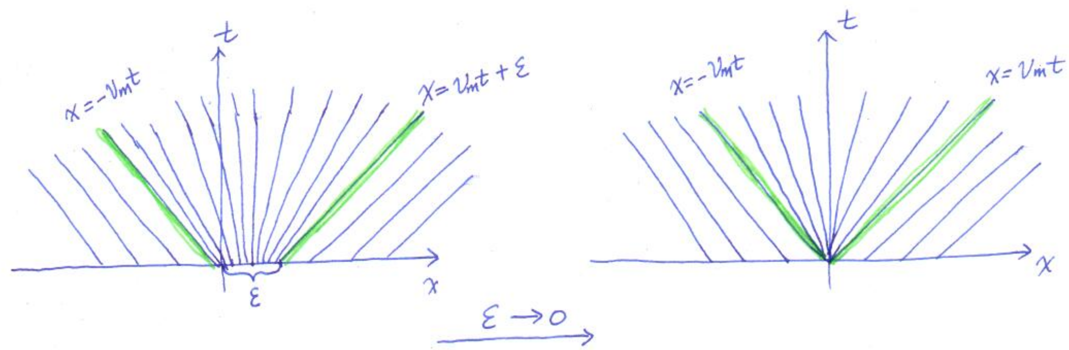

FIGURE 4. Rarefactionwave

Remark 5.1. For general equations, the self-similar solution also exists. More precisely, for the following 1-d conservation law

$$u_t + [q(u)]_x = 0,$$

if we put the ansatz $u(x,t) = \bar{u}(\xi) = \bar{u}(x/t)$, we will have

$$-\frac{x}{t^2}\bar{u}_\xi + (q(\bar{u}))_\xi \frac{1}{t} = 0, \quad \Rightarrow \bar{u}_\xi(q'(\bar{u})\xi - \xi^2) = 0, \quad \Rightarrow q'(\bar{u}(\xi)) = \xi,$$

from which the self-similar solution can be obtained directly.

In the traffic flow problem, we have $q(u) = v_m(1 - \frac{u}{\rho_m})u$ and

$$q'(u) = v_m(1 - \frac{2u}{\rho_m}), \quad \Rightarrow v_m(1 - \frac{2u(\xi)}{\rho_m}) = \xi,$$

which exactly gives the rarefaction wave solution

$$u(\xi) = \frac{1}{2}\rho_m(1 - \frac{\xi}{v_m}) = \frac{\rho_m}{2}(1 - \frac{x}{v_m t}).$$

Red light problem (or traffic jam) Initial data

$$g(x) = \begin{cases} \frac{1}{8}\rho_m, & x < 0, \\ \rho_m, & x > 0. \end{cases} \tag{5.12}$$

In this case, the wave speed is

$$q'(g(x_0)) = \begin{cases} \frac{3}{4}v_m, & x_0 < 0, \\ -v_m, & x_0 > 0. \end{cases} \tag{5.13}$$

It is obvious to see that some characteristic lines will hit together for $t > 0$. Then the main problem is how to define the solution with jump discontinuity. We will find the discontinuity curve of the solution by the integral version of the equation, i.e.,

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)dx = -q(\rho(x_2,t)) + q(\rho(x_1,t)).$$

If ρ is smooth except a line $x = s(t)$ for $t \in [t_1, t_2]$, then

$$\frac{d}{dt}\Big(\int_{x_1}^{s(t)} \rho(x,t)dx + \int_{s(t)}^{x_2} \rho(x,t)dx\Big) + q(\rho(x_2,t)) - q(\rho(x_1,t)) = 0.$$

By taking the derivative with respect to t,

$$\frac{d}{dt}\int_{x_1}^{s(t)} \rho(x,t)dx = \int_{x_1}^{s(t)} \rho_t(x,t)dx + \rho^-(s(t),t)\frac{ds}{dt},$$
$$\frac{d}{dt}\int_{s(t)}^{x_2} \rho(x,t)dx = \int_{s(t)}^{x_2} \rho_t(x,t)dx - \rho^+(s(t),t)\frac{ds}{dt},$$

where $\rho^{\pm}(s(t),t) = \lim\limits_{y\to s(t)\pm} \rho(y,t)$, we arrive at

$$\int_{x_1}^{x_2} \rho_t(x,t)dx + \Big(\rho^-(s(t),t) - \rho^+(s(t),t)\Big)\frac{ds}{dt} = q(\rho(x_1,t)) - q(\rho(x_2,t)).$$

Let $x_2 \to s(t)+$ and $x_1 \to s(t)-$, it follows that

$$\Big(\rho^-(s(t),t) - \rho^+(s(t),t)\Big)\frac{ds}{dt} = q(\rho^-(s(t),t)) - q(\rho^+(s(t),t)).$$

Therefore,

$$\frac{ds}{dt} = \frac{q(\rho^+(s(t),t)) - q(\rho^-(s(t),t))}{\rho^+(s(t),t) - \rho^-(s(t),t)} =: \frac{[q(\rho)]}{[\rho]}. \tag{5.14}$$

which is called the **Rankine-Hugoniot condition**.

Usually, we call the discontinuity propagating of solution **shock wave**.

Now in the traffic flow problem with red light initial data (5.12), $\rho^+ = \rho_m$, $\rho^- = \frac{\rho_m}{8}$, $q(\rho^+) = 0$, $q(\rho^-) = \frac{7}{64}v_m\rho_m$, so

$$\frac{ds}{dt} = \frac{q(\rho^+) - q(\rho^-)}{\rho^+ - \rho^-} = -\frac{1}{8}v_m.$$

Therefore the solution is given by

$$\rho(x,t) = \begin{cases} \frac{1}{8}\rho_m, & x < -\frac{1}{8}v_m t, \\ \rho_m, & x > -\frac{1}{8}v_m t. \end{cases} \tag{5.15}$$

5.2.2. *Burger's equation.* Riemann problem of Burger's equation,

$$u_t + uu_x = 0, \quad u|_{t=0} = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0. \end{cases}$$

The difference between the traffic flow equation and the Burger's equation is the flux function $q(u)$, which is concave in traffic flow but convex in Burger's equation.

Then the situations to have rarefaction wave and shock wave in Burger's equation are exactly opposite to those in traffic flow problem. More precisely, we will have rarefaction wave when $u_l < u_r$ and shock wave when $u_l > u_r$.

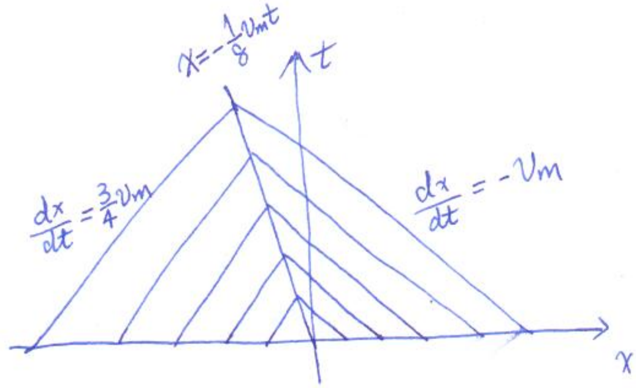

FIGURE 5. Shockwave

In the case of $u_l < u_r$, the rarefaction wave solution is given by

$$u(x,t) = \begin{cases} u_l, & x < u_l t, \\ x/t, & u_l t \le x \le u_r t, \\ u_r & x > u_r. \end{cases}$$

In the case of $u_l > u_r$, the shock wave solution is

$$u(x,t) = \begin{cases} u_l, & x < \frac{u_l+u_r}{2}t, \\ u_r, & x > \frac{u_l+u_r}{2}t. \end{cases}$$

5.3. ***Weak Entropy Solution.** In general, we cannot expect global existed smooth solution for conservation laws, no matter how smooth the initial data is. The only possible case to have a global classical solution is that the characteristics don't interact each other. Therefore, a weaker definition of the solution is expected, so that we can still solve the problem in a weaker sense.

We give the definition of weak solution in the sense of distribution

Definition 17. If u is a bounded function defined on $\mathbb{R} \times [0, +\infty)$, and $\forall v \in C_0^\infty(\mathbb{R} \times [0, +\infty))$, the following holds

$$\int_0^{+\infty} \int_{\mathbb{R}} (uv_t + q(u)v_x) dx dt + \int_{\mathbb{R}} g(x)v(x,0)dx = 0, \tag{5.16}$$

then u is called a weak solution of (5.3).

With this definition, a series of questions arise

- If u has a discontinuity, is it determined by the weak formula? (Answer: R-H condition)
- Is the weak solution unique? (Answer: No.)
- If it is not unique, how to choose the physically correct solution? (Answer: Entropy condition)

5.3.1. *Rankine-Hugoniot condition.* We answer the first question. Let V be an open subset in $\mathbb{R} \times [0, +\infty)$. If u is piecewise smooth in $\bar{V} = \bar{V}^+ \cup \bar{V}^-$ and $u \in C^1(\bar{V}^+)$, $u \in C^1(\bar{V}^-)$. Let $\Gamma = \bar{V}^+ \cap \bar{V}^-$,

$$u^{\pm}(x_0, t_0) = \lim_{(y,t) \in V^{\pm} \to (x_0, t_0)} u(y, t), \qquad \forall (x_0, t_0) \in \Gamma.$$

We assume that $u^+ \neq u^-$. Then by the definition of weak solution, $\forall v \in C_0^\infty(K)$ where K is a compact subset in V, we have

$$\begin{aligned}
0 &= \int_0^{+\infty} \int_{\mathbb{R}} [u v_t + q(u) v_x] dx dt \\
&= \int_{V^+} [u v_t + q(u) v_x] dx dt + \int_{V^-} [u v_t + q(u) v_x] dx dt \\
&= -\int_{V^+} [u_t + (q(u))_x] v dx dt - \int_{V^-} [u_t + (q(u))_x] v dx dt \\
&\quad + \int_\Gamma v(u^+, q(u^+))^T \cdot \gamma dl - \int_\Gamma v(u^-, q(u^-))^T \cdot \gamma dl,
\end{aligned}$$

where γ is the unit outer normal vector of V^+ on Γ. Then by the arbitrariness of v, we have

$$(u^+ - u^-, q(u^+) - q(u^-))^T \cdot \gamma = 0 \quad \text{along } \Gamma,$$

which means that the discontinuity of u, which is Γ, is a curve whose slope is given by

$$\dot{s}(t) = \frac{q(u^+(s,t) - q(u^-(s,t))}{u^+(s,t) - u^-(s,t)}.$$

This is called the Rankine-Hugoniot condition for shock curve Γ. The R-H condition for jump discontinuity is usually abbreviated by

$$\dot{s}(t) = \frac{[q(u)]}{[u]}. \tag{5.17}$$

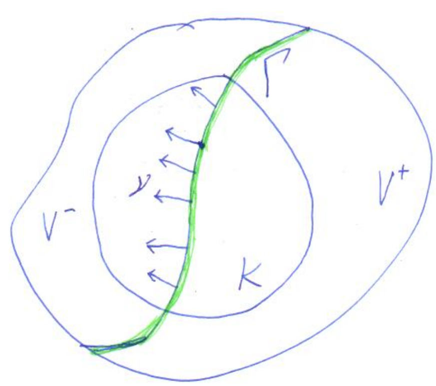

FIGURE 6. Domains in the integral in getting R-H condition

5.3.2. *Nonuniqueness of weak solution.* We will use Burger's equation to give an example,

$$u_t + uu_x = 0, \quad u|_{t=0} = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

We know already that this problem has a rarefaction wave solution

$$u(x,t) = \begin{cases} 0, & x \leq 0, \\ x/t, & 0 < x < t, \\ 1, & x \geq t. \end{cases}$$

However this problem has another weak solution (shock solution) which also satisfies the R-H condition (5.17).

$$w(x,t) = \begin{cases} 0, & x < t/2, \\ 1, & x > t/2. \end{cases}$$

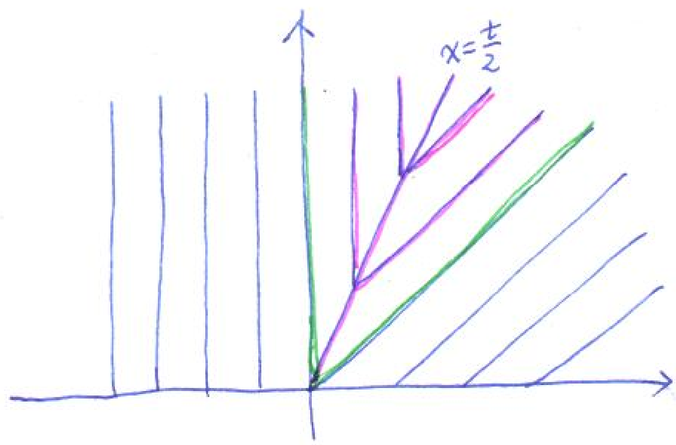

FIGURE 7. Nonuniqueness of weak solution, example 1

In fact the situation is even worse, for Burger's equation with initial data

$$u|_{t=0} = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0, \end{cases} \qquad u_l < u_r.$$

There is a family of infinite number of weak solutions, i.e., $\forall u_m \in [u_-, u_+]$, $s_m = \frac{u_- + u_m}{2}$,

$$u(x,t) = \begin{cases} u_-, & x \leq s_m t, \\ u_m, & s_m t \leq x \leq u_m t, \\ x/t, & u_m t \leq x \leq u_+ t, \\ u_+, & x > u_+ t. \end{cases} \tag{5.18}$$

Definition 18. Lax Entropy condition Let $q'' > 0$, a discontinuity propagating with speed $\dot{s}(t)$ given by the Rankine-Hugoniot jump condition satisfies the Lax entropy condition if

$$q'(u_+) < \dot{s}(t) < q'(u_-).$$

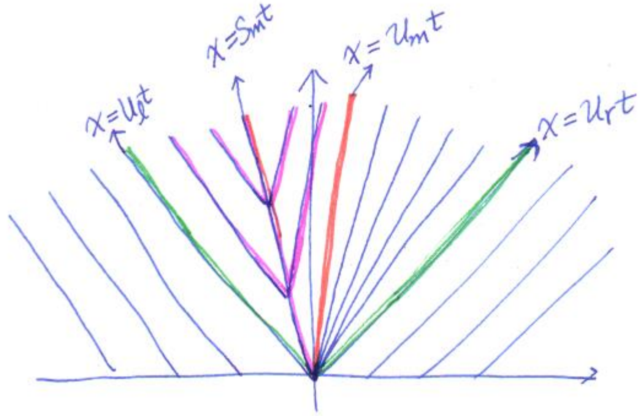

FIGURE 8. Nonuniqueness of weak solution, example 2

Lax Entropy condition says that $\forall t > 0$ once the solution jumps from x_- to x_+, the value of the solution $u(x,t)$ can only decrease, i.e., $u(x_-,t) > u(x_+,t)$. The weak solutions from (5.18) are not allowed. The similar situation holds for the case $q'' > 0$ as well.

We give the following existence and uniqueness of weak entropy solution without proof.

Theorem 5.3. *If $q \in C^2(\mathbb{R})$ is convex (or concave), g is bounded, there exists a unique weak entropy solution of*

$$\begin{cases} u_t + (q(u))_x = 0, & x \in \mathbb{R}, t > 0, \\ u|_{t=0} = g(x), & x \in \mathbb{R}. \end{cases}$$

We refer the proof of this theorem to J. Smoller's book.

5.3.3. *Riemann problem for general scalar conservation law.* As a summary, we give the entropy solution of Riemann problems to general scalar conservation law

$$u_t + (q(u))_x = 0, \quad x \in \mathbb{R}, t > 0,$$
$$u|_{t=0} = \begin{cases} u_+, & x > 0, \\ u_-, & x < 0. \end{cases}$$

where $u_+ \neq u_-$ are constants.

Theorem 5.4. *If $q \in C^2(\mathbb{R})$ is strictly convex and $q'' \geq h > 0$, we have the following result*

- *If $u_+ < u_-$, then the unique entropy solution is shock wave, i.e.,*

$$u(x,t) = \begin{cases} u_+, & x/t > \dot{s}(t), \\ u_-, & x/t < \dot{s}(t), \end{cases}$$

where

$$\dot{s}(t) = \frac{q(u_+) - q(u_-)}{u_+ - u_-}.$$

- *If $u_+ > u_-$, then the unique entropy solution is the rarefaction wave, i.e.,*

$$u(x,t) = \begin{cases} u_+, & x/t > q'(u_+), \\ r(x/t), & q'(u_-) < x/t < q'(u_+), \\ u_-, & x/t < q'(u_-), \end{cases}$$

where $r = (q')^{-1}$.

Remark 5.2. The result is similar in the concave flux case for $q'' \le h < 0$.

5.4. **Viscous Burger's equation.** Vanishing viscosity method is natural from physical point of view to study discontinuous solutions for hyperbolic equations. In fluid dynamics, Navier-Stokes system is a viscous version of Euler system. Roughly speaking, vanishing viscosity is to use $u_t + q(u) = \varepsilon u_{xx}$ as an approximation of $u_t + q(u) = 0$, then take $\varepsilon \to 0$, one could get desired properties of solution. Usually, parabolic equations are easier, and the solutions have better properties.

We will focus on the viscous Burger's equation in this subsection.

5.4.1. *Cole Hopf transformation in 1950's.* In 1950's Cole and Hopf found a transformation independently to reduce the viscous Burger's equation into a heat equation. This transformation is now called Cole-Hopf transformation. Then by using the fundamental solution of heat equation, an explicit solution of viscous Burger's equation can be obtained.

Viscous Burger's equation is

$$u_t + uu_x = \varepsilon u_{xx}, \qquad x \in \mathbb{R}, t > 0.$$

which can be rewritten into

$$u_t + \left(\frac{1}{2}u^2 - \varepsilon u_x\right)_x = 0.$$

This formula means that the $2-D$ vector valued function $(-u, \frac{1}{2}u^2 - \varepsilon u_x)$ is curl free. Therefore, there exists a potential $\psi(x,t)$ such that

$$\psi_x = -u, \qquad \psi_t = \frac{1}{2}u^2 - \varepsilon u_x.$$

So ψ solves the equation

$$\psi_t = \frac{1}{2}\psi_x^2 + \varepsilon\psi_{xx}.$$

A way to avoid the quadratic term ψ_x^2 is to use a transformation $\psi = g(\varphi)$ with g to be determined later, so that

$$\psi_t = g'(\varphi)\varphi_t, \quad \psi_x = g'(\varphi)\varphi_x, \quad \psi_{xx} = g''(\varphi)(\varphi_x)^2 + g'(\varphi)\varphi_{xx}.$$

Then the equation that φ satisfies is

$$g'(\varphi)[\varphi_t - \varepsilon\varphi_{xx}] = [\frac{1}{2}(g'(\varphi))^2 + \varepsilon g''(\varphi)](\varphi_x)^2.$$

The function g can be chosen such that the right hand side vanishes. For example, by taking $g(s) = 2\varepsilon \log s$, the equation for φ is reduced to the heat equation,

$$\varphi_t - \varepsilon\varphi_{xx} = 0.$$

The relation between u and φ is

$$u = -\psi_x = -2\varepsilon \frac{\varphi_x}{\varphi},$$

which is called the **Cole-Hopf transformation**.

The initial data for $u(x, 0) = u_0(x)$ is transformed into

$$\varphi_0(x) = \exp\Big\{ - \int_a^x \frac{u_0(z)}{2\varepsilon} dz \Big\}, \quad a \in \mathbb{R}.$$

If

$$\frac{1}{x^2} \int_a^x u_0(z) dz \to 0, \qquad \text{as } |x| \to \infty,$$

Then the Cauchy problem for φ has a unique smooth solution,

$$\varphi(x,t) = \frac{1}{\sqrt{4\pi\varepsilon t}} \int_{-\infty}^{+\infty} \varphi_0(y) \exp\Big\{ - \frac{(x-y)^2}{4\varepsilon t} \Big\} dy.$$

Changing back to the original variables, we know that the Cauchy problem for viscous Burger's equation has solution

$$u(x,t) = \frac{\displaystyle\int_{-\infty}^{+\infty} \frac{x-y}{t} \varphi_0(y) \exp\Big\{ - \frac{(x-y)^2}{4\varepsilon t} \Big\} dy}{\displaystyle\int_{-\infty}^{+\infty} \varphi_0(y) \exp\Big\{ - \frac{(x-y)^2}{4\varepsilon t} \Big\} dy}.$$

5.4.2. *traveling wave solution of viscous Burger's equation.* The viscous Burger's equation is

$$u_t + [q(u)]_x = \varepsilon u_{xx}. \tag{5.19}$$
$$u|_{t=0} = 0.$$

In case $\varepsilon = 0$, we know that the solution has form $u(x,t) = u_0(x - q't)$ which is of the traveling wave form, i.e., u is a function of $x - q't$. For the viscous case, with the help of the diffusion, we are interested to know where there exists a traveling wave solution which can connect two constant states. Furthermore, is it related to the solution in the inviscid case?

The constant states are given via

$$\lim_{x \to -\infty} u(x,t) = u_L, \qquad \lim_{x \to +\infty} u(x,t) = u_R, \qquad u_L \neq u_R.$$

We are searching for a typical type of solution $u(x,t) = U(x - vt)$ with undetermined wave speed v. Let $\xi = x - vt$, then the ODE for U is

$$(q'(U) - v)U' = \varepsilon U'', \quad U(-\infty) = u_L, \quad U(+\infty) = u_R.$$

In addition, we assume $U' \to 0$ as $\xi \to \pm\infty$. Then integrating the equation once, we have

$$q(U) - vU + A = \varepsilon U', \quad q(u_L) - vu_L + A = 0, \quad q(u_R) - vu_R + A = 0.$$

Thus a consequence for wave speed v is that

$$v = \frac{q(u_R) - q(u_L)}{u_R - u_L}, \qquad A = \frac{-q(u_R)u_L + q(u_L)u_R}{u_R - u_L}.$$

So if a traveling wave solution exists, the wave speeding is exactly the shock speed in Rankine-Hugoniot condition.

Now we are going to check whether the traveling wave solution exists. Back to the equation for

$$\varepsilon U' = q(U) - vU + A.$$

We know that this equation has two equilibria $U = u_R$ and $U = u_L$. If $q'' < 0$, from ODE theory, we know that only the case $u_L < u_R$ can be connected. If $q'' > 0$, then $u_L > u_R$.

5.5. **Problems.**

(1) Solve Burger's equation $u_t + uu_x = 0$ with different initial data

$$g_1(x) = \begin{cases} 1, & x \le 0, \\ 1 - x, & 0 < x < 1, \\ 0, & x \ge 1, \end{cases}$$

and

$$g_2(x) = \begin{cases} 0, & x \le 0, \\ 1, & 0 < x < 1, \\ 0, & x \ge 1. \end{cases}$$

(2) If $g \in C^1(\mathbb{R})$ has compact support, prove that

$$\begin{cases} \rho_t + v_m(1 - \dfrac{2\rho}{\rho_m})\rho_x = 0, & (x,t) \in \mathbb{R} \times (0, +\infty) \\ \rho(x,0) = g(x), \end{cases}$$

has a locally existed C^1 solution.

(3) Show that, for every $\alpha \ge 1$, the function

$$u_\alpha(x,t) = \begin{cases} 1, & 2x < (1-\alpha)t, \\ -\alpha, & (1-\alpha)t < 2x < 0, \\ \alpha, & 0 < 2x < (\alpha - 1)t, \\ -1, & (\alpha - 1)t < 2x \end{cases}$$

is a weak solution of the problem

$$\begin{cases} u_t + uu_x = 0, & t > 0, x \in \mathbb{R}, \\ u(x,0) = \begin{cases} 1, & x < 0, \\ -1, & x > 0. \end{cases} \end{cases}$$

Is it also an entropy solution, at least for some α?

(4) (Traffic flow in tunnel found by Greenberg in 1959) Study the traffic flow problem with flux

$$q(\rho) = v_m \rho \log \frac{\rho_m}{\rho},$$

and initial data

$$\rho(x,0) = \begin{cases} \rho_l, & x \leq 0, \\ \rho_r, & x > 0, \end{cases}$$

where v_m is the maximum velocity, ρ_m is the maximum density and $\rho_l = \frac{1}{2}\rho_m$, $\rho_r = \frac{1}{3}\rho_m$. Give the solution, and draw a picture for partial path of car trajectories.

(5) Find the solutions of

$$\begin{cases} u_t \pm uu_x = 0, & t > 0, x \in \mathbb{R}, \\ u(x,0) = x, & x \in \mathbb{R}. \end{cases}$$

(6) Draw the characteristics and describe the evolution for $t \to +\infty$ of the solution of the problem

$$\begin{cases} u_t + uu_x = 0, & t > 0, x \in \mathbb{R}, \\ u(x,0) = \begin{cases} \sin x, & 0 < x < \pi, \\ 0 & x \leq 0 \text{ or } x \geq \pi. \end{cases} \end{cases}$$

6. ***Mean field equation

This part of the notes comes from Golse's lecture notes in 2013.

6.1. Mean field particle model.

We start with the N-particle system

$$\frac{d}{dt}x_i(t) = \frac{1}{N}\sum_{j=1}^{N} K(x_i(t), x_j(t)), \quad 1 \le i \le N,$$

where the given interaction force $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$K(x, y) = -K(y, x) \text{ and } K(x, x) = 0 \quad x, y \in \mathbb{R}^d.$$

The main features of mean field model are that the particles are not distinguishable and the pair interaction forces are of order $\frac{1}{N}$. When the number of the particles is very large, one expects to derive an effective one particle model to decribe the whole dynamics, which is called the mean filed model.

Within this chapter, we assume that the interaction force has regularity in the following sense

Assumption for K: $K \in C^1(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d)$ and there exists an $L > 0$ such that

$$\sup_y |\nabla_x K(x, y)| + \sup_x |\nabla_y K(x, y)| \le L. \tag{6.1}$$

Definition 19 (Empirical measure)**.** Fur each N-tuple $X_N := (x_1, \cdots, x_N)$ the corresponding empirical measure is given by

$$\mu_{X_N} = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}.$$

With the empirical measure, we can rewrite the interaction summation into

$$\begin{aligned} &\frac{1}{N}\sum_{i=j}^{N} K(x_i(t), x_j(t)) = \frac{1}{N}\sum_{i=j}^{N} \langle K(x_i(t), \cdot), \delta_{x_j(t)}\rangle \\ = \quad & \langle K(x_i(t), \cdot), \mu_{X_N(t)}\rangle =: \mathcal{K}\mu_{X_N(t)}(x_i(t)). \end{aligned}$$

If one can show that $\mu_{X_N(t)}$ converges to a probability measure $\mu(t)$, then the limit of the N-particle system is expected to be

$$\frac{d}{dt}x(t) = \int_{\mathbb{R}^d} K(x(t), y)\, d\mu(dy, t),$$

with given initial datum

$$x(0) = x \in \mathbb{R}^d \quad \text{and} \quad \mu(0) = \mu_0 \text{ a given probability measure.}$$

Within the whole setting, $\mu(\cdot, t)$ is the push-forward measure [1] of μ_0 along the dynamics $x(t)$.

[1] Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces and $\phi : (X, \mathcal{A}) \to (Y, \mathcal{B})$ be a measurable map, μ be a measure on $(X, \mathcal{A})$, then the push-forward measure $\phi_\sharp\mu$ is given by $\phi_\sharp\mu(B) = \mu(\phi^{-1}(B))$, $\forall B \in \mathcal{B}$.

Theorem 6.1. *Under the assumptions on K in (6.1), the problem*

$$\begin{aligned} &\frac{d}{dt}x_i(t) = \frac{1}{N}\sum_{j=1}^N K(x_i(t), x_j(t)), \quad 1 \le i \le N, \\ &x_i(t) = x_{i,0}. \end{aligned} \tag{6.2}$$

has a unique classical solution $X_N(t, X_{N,0}) \in C^1(\mathbb{R}, \mathbb{R}^{dN})$, $1 \le i \le N$. Furthermore, the empirical measure $\mu_{X_N(t,X_{N,0})}$ is a weak solution of the following Cauchy problem in the sense of distribution

$$\begin{aligned} &\partial_t \mu + \nabla \cdot (\mu \mathcal{K}\mu) = 0, \\ &\mu|_{t=0} = \mu_{X_{N,0}}. \end{aligned} \tag{6.3}$$

Proof. Due to the fact that K is uniform Lipschitz continuous, the C^1 solution of ODE system (6.2) can be obtained by nonlinear ODE theory. Next we check that the corresponding empirical measure is a weak solution of the mean field PDE (6.3).

For arbitrary test function $\varphi \in C_0^\infty(\mathbb{R}^d; \mathbb{R})$, we have

$$\begin{aligned}
\frac{d}{dt}\langle \mu_{X_N(t,X_{N,0})}, \varphi\rangle &= \frac{d}{dt}\Big\langle \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t,X_{N,0})}, \varphi\Big\rangle \\
&= \frac{d}{dt}\frac{1}{N}\sum_{i=1}^N \varphi(x_i(t, X_{N,0})) \\
&= \frac{1}{N}\sum_{i=1}^N \nabla\varphi(x_i(t, X_{N,0}))\frac{d}{dt}x_i(t, X_{N,0}) \\
&= \frac{1}{N}\sum_{i=1}^N \nabla\varphi(x_i(t, X_{N,0}))\frac{1}{N}\sum_{j=1}^N K(x_i(t, X_{N,0}), x_j(t, X_{N,0})) \\
&= \frac{1}{N}\sum_{i=1}^N \nabla\varphi(x_i(t, X_{N,0}))\frac{1}{N}\sum_{j=1}^N \Big\langle \delta_{x_j(t,X_{N,0})}, K(x_i(t, X_{N,0}), \cdot)\Big\rangle \\
&= \frac{1}{N}\sum_{i=1}^N \nabla\varphi(x_i(t, X_{N,0}))\Big\langle \mu_{X_N(t,X_{N,0})}, K(x_i(t, X_{N,0}), \cdot)\Big\rangle \\
&= \frac{1}{N}\sum_{i=1}^N \nabla\varphi(x_i(t, X_{N,0}))\mathcal{K}\mu_{X_N(t,X_{N,0})}(x_i(t, X_{N,0})) \\
&= \Big\langle \mu_{X_N(t,X_{N,0})}, \nabla\varphi\mathcal{K}\mu_{X_N(t,X_{N,0})}\Big\rangle \\
&= \Big\langle \mathcal{K}\mu_{X_N(t,X_{N,0})}\mu_{X_N(t,X_{N,0})}, \nabla\varphi\Big\rangle \\
&= -\Big\langle \nabla\Big(\mathcal{K}\mu_{X_N(t,X_{N,0})}\mu_{X_N(t,X_{N,0})}\Big), \varphi\Big\rangle.
\end{aligned}$$

Notice that K is continuously differentiable, we have that $\mathcal{K}\mu_{X_N(t,X_{N,0})}$ is also continuously differential. Hence $\mathcal{K}\mu_{X_N(t,X_{N,0})}\mu_{X_N(t,X_{N,0})}$ is a distribution. Therefore $\mu_{X_N(t,X_{N,0})}$ is a solution of (6.3) in the sense of distribution. $\square$

6.2. **Solvability of the mean field equation.** In this subsection, we give the unique solvability of the mean field equation with general initial data. More precisely, we consider the following problem

$$\begin{aligned}
&\partial_t \mu + \nabla \cdot (\mu \mathcal{K}\mu) = 0, \\
&\mu|_{t=0} = \mu_0.
\end{aligned} \tag{6.4}$$

The corresponding characteristics of this PDE is given by

$$\begin{aligned}
&\frac{d}{dt}x(t,x_0,\mu_0) = \int_{\mathbb{R}^d} K(x(t,x_0,\mu_0),y)\mu(dy,t), \\
&x(0,x_0,\mu_0) = x_0, \quad \forall x_0 \in \mathbb{R}^d, \\
&\mu(\cdot,t) = x(t,\cdot,\mu_0)_\sharp \mu_0.
\end{aligned} \tag{6.5}$$

The solution $\mu(\cdot,t)$ is searched in the space

$$\mathcal{P}_1(\mathbb{R}^d) = \{\mu \in (\mathbb{R}^d) | \int_{\mathbb{R}^d} |x|\mu(dx) < \infty\}, \tag{6.6}$$

where $\mathcal{P}(\mathbb{R}^d)$ is the space of all probability measures.

Theorem 6.2. *Let the assumtion for K in* (6.1) *holds, $\mu_0 \in \mathcal{P}_1$, then problem* (6.5) *has a unique solution $x(t,x_0,\mu_0) \in C^1(\mathbb{R};\mathbb{R}^d)$ and $x(t,\cdot,\mu_0)_\sharp\mu_0 \in \mathcal{P}_1$, $\forall t > 0$.*

Proof. The proof is based on Picard iteration. Let $C_1 = \int_{\mathbb{R}^d} |x|\mu_0(dx)$ and define the following space

$$X := \Big\{ v \in C(\mathbb{R}^d;\mathbb{R}^d) \Big| \sup_{x\in\mathbb{R}^d} \frac{|v(x)|}{1+|x|} < \infty \Big\}. \tag{6.7}$$

It is obvious that X is a Banach space with norm

$$\|v\|_X := \sup_{x\in\mathbb{R}^d} \frac{|v(x)|}{1+|x|}.$$

As preparations, we need the following estimates for the nonlocal term by using the assumption (6.1) for K. $\forall v, w \in X$ we have

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^d} K(v(x),v(y))\mu_0(dy) - \int_{\mathbb{R}^d} K(w(x),w(y))\mu_0(dy) \Big| \\
\le\ & L \int_{\mathbb{R}^d} (|v(x)-w(x)| + |v(y)-w(y)|)\mu_0(dy) \\
\le\ & L\|v-w\|_X(1+|x|) + L\|v-w\|_X \int_{\mathbb{R}^d} (1+|y|)\mu_0(dy) \\
\le\ & L(2+C_1)\|v-w\|_X(1+|x|).
\end{aligned} \tag{6.8}$$

We define the following sequence by induction

$$
\begin{aligned}
x_0(t,y) &= y, \quad \forall y \in \mathbb{R}^d, \\
x_n(t,y) &= y + \int_0^t \int_{\mathbb{R}^d} K(x_{n-1}(s,y), x_{n-1}(s,z))\mu_0(dz)ds.
\end{aligned}
$$

Then the difference between x_1 and x_0 can be given by

$$
\begin{aligned}
|x_1(t,y) - x_0(t,y)| &= \Big| \int_0^t \int_{\mathbb{R}^d} K(x_0(s,y), x_0(s,z))\mu_0(dz)ds \Big| \\
&= \Big| \int_0^t \int_{\mathbb{R}^d} K(y,z)\mu_0(dz)ds \Big| \\
&\leq \int_0^{|t|} \int_{\mathbb{R}^d} L(|y| + |z|)\mu_0(dz)ds \\
&= \int_0^{|t|} L(|y| + C_1)ds \leq L(1+C_1)(1+|y|)|t|.
\end{aligned}
$$

Furthermore $\forall n \geq 1$, we have

$$
\begin{aligned}
& |x_n(t,y) - x_{n-1}(t,y)| \\
= & \Big| \int_0^t \int_{\mathbb{R}^d} \Big(K(x_{n-1}(s,y), x_{n-1}(s,z)) - K(x_{n-2}(s,y), x_{n-2}(s,z)) \Big) \mu_0(dz)ds \Big| \\
\leq & L(2+C_1) \int_0^{|t|} \|x_{n-1}(x,\cdot) - x_{n-2}(s,\cdot)\|_X (1+|y|)ds,
\end{aligned}
$$

hence by deviding both sides by $1+|y|$, we have

$$
\|x_n(t,\cdot) - x_{n-1}(t,\cdot)\|_X \leq L(2+C_1) \int_0^{|t|} \|x_{n-1}(x,\cdot) - x_{n-2}(s,\cdot)\|_X ds \leq \frac{((2+C_1)L|t|)^n}{(n-1)!},
$$

which implies $\forall n > m \to \infty$,

$$
\|x_n(t,\cdot) - x_m(t,\cdot)\|_X \leq \sum_{i=m}^{n-1} \|x_{i+1}(t,\cdot) - x_i(t,\cdot)\|_X \to 0.
$$

Therefore for $T > 0$,

$$
x_n(t,\cdot) \to x(t,\cdot) \text{ in } X \text{ uniformly in } [-T,T],
$$

and $x \in C(\mathbb{R};\mathbb{R}^d)$ satisfies $\forall y \in \mathbb{R}^d$

$$
x(t,y) = y + \int_0^t \int_{\mathbb{R}^d} K(x(s,y), x(s,z))\mu_0(dz)ds.
$$

According to the fundamental theorem in calculus, we know that for $y \in \mathbb{R}^d$, $x(t,y) = C^1(\mathbb{R};\mathbb{R}^d)$ and it satisfiles

$$
\frac{d}{dt}x(t,y) = \int_{\mathbb{R}^d} K(x(t,y), x(t,z))\mu_0(dz) = \int_{\mathbb{R}^d} K(x(t,y), z')\mu(dz',t)
$$

where $\mu(\cdot,t)$ is the push forward measure of μ_0 along $x(t,\cdot)$.

The uniqueness will be obtained in the following. Let $x, \bar{x}$ be two solutions, then by taking the difference we have

$$x(t,y) - \bar{x}(t,y) = \int_0^t \int_{\mathbb{R}^d} \Big(K(x(s,y), x(s,z)) - K(\bar{x}(s,y), \bar{x}(s,z)) \Big) \mu_0(dz) ds.$$

With the help of similar estimates as before, we obtain

$$\|x(t,\cdot) - \bar{x}(t,\cdot)\|_X \le L(2 + C_1) \int_0^{|t|} \|x(s,\cdot) - \bar{x}(s,\cdot)\|_X ds.$$

This implies that $\|x(t,\cdot) - \bar{x}(t,\cdot)\|_X = 0$ by applying Gronwall's inequality and initial data $\|x(0,\cdot) - \bar{x}(0,\cdot)\|_X = 0$. $\square$

6.3. **Mean field limit (stability).** We will prove the stability of mean field PDE by using the so called Monge-Kantorovich distance (or Wasserstein distance)

Definition 20. For two measures $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$ $(p \ge 1)$ with

$$\mathcal{P}_p(\mathbb{R}^d) = \{\mu \in \mathcal{P}(\mathbb{R}^d) | \int_{\mathbb{R}^d} |x|^p \mu(dx) < \infty\},$$

the Monge-Kantonovich distance $\mathrm{dist}_{MK,p}(\mu,\nu)$ or $W^p(\mu,\nu)$ is defined by

$$\mathrm{dist}_{MK,p}(\mu,\nu) = W^p(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} \Big(\iint_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^p \pi(dxdy) \Big)^{\frac{1}{p}} \tag{6.9}$$

where

$$\Pi(\mu,\nu) = \Big\{ \pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \Big| \int_{\mathbb{R}^d} \pi(\cdot, dy) = \mu(\cdot) \text{ and } \int_{\mathbb{R}^d} \pi(dx, \cdot) = \nu(\cdot) \Big\}.$$

Remark 6.1. $\forall \varphi, \psi \in C(\mathbb{R}^d)$ such that $\varphi(x) \sim O(|x|^p)$ for $|x| \gg 1$ and $\psi(y) \sim O(|y|^p)$ for $|y| \gg 1$, for $\pi \in \Pi(\mu,\nu)$ it holds

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} (\varphi(x) + \psi(y)) \pi(dxdy) = \int_{\mathbb{R}^d} \varphi(x) \mu(dx) + \int_{\mathbb{R}^d} \psi(y) \nu(dy).$$

Remark 6.2. It can be proved that the W^1 distance can be computed by

$$\mathrm{dist}_{MK,1}(\mu,\nu) = W^1(\mu,\nu) = \sup_{\phi \in Lip(\mathbb{R}^d), Lip(\phi) \le 1} \Big| \int_{\mathbb{R}^d} \phi(x) \mu(dx) - \int_{R^d} \phi(x) \nu(dx) \Big|.$$

Theorem 6.3 (Dobrushin's stability). *Let $\mu_0, \bar{\mu}_0 \in \mathcal{P}_1(\mathbb{R}^d)$ and $(x(t,\cdot,\mu_0), \mu(\cdot,t))$, $(x(t,\cdot,\bar{\mu}_0), \bar{\mu}(\cdot,t))$ be solutions of* (6.5). *Then $\forall t > 0$, it holds*

$$\mathrm{dist}_{MK,1}(\mu(\cdot,t), \bar{\mu}(\cdot,t)) \le e^{2|t|L} \mathrm{dist}_{MK,1}(\mu_0, \bar{\mu}_0).$$

Proof. Let (x_0, μ_0) and $(\bar{x}_0, \bar{\mu}_0)$ be two initial data of problem (6.5) and $\pi_0 \in \Pi(\mu_0, \bar{\mu}_0)$, taking the difference of these two problems, we have

$$\begin{aligned} & x(t, x_0, \mu_0) - x(t, \bar{x}_0, \bar{\mu}_0) \\ = \quad & x_0 - \bar{x}_0 + \int_0^t \int_{\mathbb{R}^d} K(x(s, x_0, \mu_0), y) \mu(s, dy) ds - \int_0^t \int_{\mathbb{R}^d} K(x(s, \bar{x}_0, \bar{\mu}_0), y) \bar{\mu}(s, dy) ds, \end{aligned}$$

where $\mu(\cdot,t) = x(t,\cdot,\mu_0)_\sharp\mu_0$ and $\bar\mu(\cdot,t) = x(t,\cdot,\bar\mu_0)_\sharp\bar\mu_0$. Now we compute further and get

$$\begin{aligned}
& x(t,x_0,\mu_0) - x(t,\bar x_0,\bar\mu_0) \\
= \;& x_0 - \bar x_0 + \int_0^t \int_{\mathbb{R}^d} K(x(s,x_0,\mu_0), x(s,z,\mu_0))\mu_0(dz)ds \\
& \qquad - \int_0^t \int_{\mathbb{R}^d} K(x(s,\bar x_0,\bar\mu_0), x(s,\bar z,\bar\mu_0))\bar\mu_0(d\bar z)ds \\
= \;& x_0 - \bar x_0 + \int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} \Big(K(x(s,x_0,\mu_0), x(s,z,\mu_0)) \\
& \qquad -K(x(s,\bar x_0,\bar\mu_0), x(s,\bar z,\bar\mu_0))\Big)\pi_0(dzd\bar z)ds.
\end{aligned}$$

Therefore by assumption (6.1) for K, we have

$$\begin{aligned}
& |x(t,x_0,\mu_0) - x(t,\bar x_0,\bar\mu_0)| \le |x_0 - \bar x_0| \\
& \quad +L\int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} \Big(|x(s,x_0,\mu_0) - x(s,\bar x_0,\bar\mu_0)| + |x(s,z,\mu_0) - x(s,\bar z,\bar\mu_0)|\Big)\pi_0(dzd\bar z)ds \\
& \le |x_0 - \bar x_0| + L\int_0^t |x(s,x_0,\mu_0) - x(s,\bar x_0,\bar\mu_0)|ds \\
& \quad +L\int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x(s,z,\mu_0) - x(s,\bar z,\bar\mu_0)|\pi_0(dzd\bar z)ds
\end{aligned}$$

Next we integrate both sides in $x_0, \bar x_0$ with respect to measure π_0,

$$\begin{aligned}
\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x(t,x_0,\mu_0) - x(t,\bar x_0,\bar\mu_0)|\pi_0(dx_0d\bar x_0) &\le \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x_0 - \bar x_0|\pi_0(dx_0d\bar x_0) \\
& +L\int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x(s,x_0,\mu_0) - x(s,\bar x_0,\bar\mu_0)|\pi_0(dx_0d\bar x_0)ds \\
& +L\int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x(s,z,\mu_0) - x(s,\bar z,\bar\mu_0)|\pi_0(dzd\bar z)ds.
\end{aligned}$$

By denoting

$$D[\pi_0](t) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x(s,z,\mu_0) - x(s,\bar z,\bar\mu_0)|\pi_0(dzd\bar z),$$

we have obtained the estimate

$$D[\pi_0](t) \le D[\pi_0](0) + 2L\int_0^t D[\pi_0](s)ds,$$

which implies by Gronwall's inequality that

$$D[\pi_0](t) \le D[\pi_0](0)e^{2Lt}.$$

Now let $\phi_t : \mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d\times\mathbb{R}^d$ be the map such that $\phi_t(x_0,\bar x_0) = (x(t,x_0,\mu_0), x(t,\bar x_0,\bar\mu_0))$ and for arbitrary $\pi_0 \in \Pi(\mu_0,\nu_0)$, $\pi_t := \phi_{t\sharp}\pi_0$ be the push forward measure of π_0 by ϕ_t. It is obvious that

$$\pi_t = \phi_{t\sharp}\pi_0 \in \Pi(\mu(\cdot,t), \bar\mu(\cdot,t)).$$

Therefore,

$$\begin{aligned}
& \mathrm{dist}_{MK,1}(\mu(\cdot,t),\bar{\mu}(\cdot,t)) \\
= \quad & \inf_{\pi\in\Pi(\mu(\cdot,t),\bar{\mu}(\cdot,t))} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |z-\bar{z}|\pi(dzd\bar{z}) \\
\leq \quad & \inf_{\pi_0\in\Pi(\mu_0,\bar{\mu}_0)} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x(t,z,\mu_0)-x(t,\bar{z},\bar{\mu}_0)|\pi_0(dzd\bar{z}) \\
= \quad & \inf_{\pi_0\in\Pi(\mu_0,\bar{\mu}_0)} D[\pi_0](t) \leq \inf_{\pi_0\in\Pi(\mu_0,\bar{\mu}_0)} D[\pi_0](0)e^{2Lt} = e^{2Lt}\mathrm{dist}_{MK,1}(\mu_0,\bar{\mu}_0).
\end{aligned}$$

□

Corollary 6.1. *Let f_0 be a probability density on $\mathbb{R}^d$ with $\int_{\mathbb{R}^d}|x|f_0(x)dx<\infty$. Then the Cauchy problem*

$$\begin{aligned}
&\partial_t f + \nabla\cdot(f\mathcal{K}f) = 0 \\
&f|_{t=0} = f_0
\end{aligned}\tag{6.10}$$

has a unique weak solution $f\in C(\mathbb{R};L^1(\mathbb{R}^d))$.

Proof. Notice that if μ_0 has a density f_0, i.e. absolutely continuous with respect to Lebesgue measure, then the push forward measure $\mu(\cdot,t)$ is also absolutely continuous with respect to Lebesgue measure, therefore, there exists an $f(x,t)\in L^1(\mathbb{R}^d)$ for any fixed t, such that $\mu(dx,t)=f(x,t)dx$. □

Theorem 6.4 (mean field limit). *For $f_0\in L^1(\mathbb{R}^d)$, let $\mu_{0,N}=\frac{1}{N}\sum_{i=1}^N\delta_{x_{i,0}}$ such that*

$$\mathrm{dist}_{MK,1}(\mu_{0,N},f_0)\to 0, \quad \text{as } N\to\infty.$$

Let $X_N(t)$ be the solution of the N particle system (6.2) *with its empiricle measure $\mu^N(t)=\mu_{X_N(t,X_{N,0})}$. Then*

$$\mathrm{dist}_{MK,1}(\mu^N(t),f(\cdot,t))\leq e^{2Lt}\mathrm{dist}_{MK,1}(\mu_{0,N},f_0)\to 0, \quad \text{as } N\to\infty.$$

And $\mu^N(t)\rightharpoonup f(\cdot,t)$ weakly in measure.

Proof. The stability result gives us already the convergence rate estimate. We are left to prove the weak convergence in measure. $\forall\phi\in Lip(\mathbb{R}^d)$, we have

$$\begin{aligned}
& \Big|\int_{\mathbb{R}^d}\phi(x)\mu^N(t,dx)-\int_{\mathbb{R}^d}\phi(x)f(t,x)dx\Big| \\
= \quad & \Big|\iint_{\mathbb{R}^d\times\mathbb{R}^d}(\phi(x)-\phi(y))\pi_t(dxdy)\Big| \qquad \text{where } \pi_t\in\Pi(\mu^N(t),f(\cdot,t)) \\
\leq \quad & Lip(\phi)\iint_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|\pi_t(dxdy)\to 0.
\end{aligned}$$

Since $Lip(\mathbb{R}^d)$ is dense in $C_c(\mathbb{R}^d)$ and because of the total mass is 1, then the argument also holds for test function in $C_b(\mathbb{R}^d)$. hence the weak convergence in measure is true. □

Lehrstuhl für Mathematik IV Universität Mannheim, D-68131 Mannheim
Email address: chen@math.uni-mannheim.de