

## LECTURE NOTES FOR INTRODUCTION TO PDES

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This is an introductory course for partial differential equations. Four basic types of PDEs, i.e. the transport equation, wave equation, heat equation, and Poisson equation, will be studied. Derivation of the solution formula by using characteristic method, separation of variable (Fourier series), and Fourier transformation are going to be introduced for specific equations, the technics can be used also for other type of equations. Classical tools such as maximum principle and energy estimates are introduced in order to get uniqueness and stability of the solutions.

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*Date:* HWS2019.

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## 1. INTRODUCTION

**1.1. Definition and examples of linear and nonlinear equations.** Within this lecture notes, we will use the following notations.

$\Omega$  is an open subset in  $\mathbb{R}^n$ . Let  $u$  be a smooth function defined on  $\Omega$ . A **multi-index** is an  $n$  dimensional vector, i.e.,  $\alpha = (\alpha_1, \dots, \alpha_n)$  and the differential operator  $D^\alpha$  is defined by

$$D^\alpha u = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_n}^{\alpha_n} u.$$

The order of the operator  $D^\alpha$  is given by  $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$ , i.e.,  $D^\alpha$  is an  $|\alpha|$ -th order differential operator.

$F$  is a mapping in the following sense

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \dots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}.$$

Symbolically a  $k$ -th order partial differential equation can be written in the form of

$$F(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x) = 0, \quad \forall x \in \Omega, \quad (1.1)$$

where  $D^k$  are all the  $k$ -th order differential operators, i.e.,  $D^\alpha$  with  $|\alpha| = k$ . The unknown in the equation is  $u(x) : \Omega \rightarrow \mathbb{R}$ .

Moreover, if  $\mathbf{F}$  is a vector-valued function such that

$$\mathbf{F} : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \dots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times \Omega \rightarrow \mathbb{R}^m,$$

then the corresponding equation is a  $k$ -th order partial differential system

$$\mathbf{F}(D^k \mathbf{u}(x), D^{k-1} \mathbf{u}(x), \dots, D\mathbf{u}(x), \mathbf{u}(x), x) = \mathbf{0}, \quad \forall x \in \Omega \quad (1.2)$$

and  $\mathbf{u} : \Omega \rightarrow \mathbb{R}^m$  is the unknown.

If the equation can be written as

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = f(x) \quad (1.3)$$

with  $a_\alpha(x)$  and  $f(x)$  being given functions, then it is called a **linear** equation.

If the equation can be written as

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u + a_0(D^{k-1} u, \dots, Du, u, x) = 0 \quad (1.4)$$

with  $a_\alpha(x)$  being a given function and  $a_0$  being a given mapping, then it is called a **semi-linear** equation.

If the equation can be written as

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1} u, \dots, Du, u, x) D^\alpha u + a_0(D^{k-1} u, \dots, Du, u, x) = 0 \quad (1.5)$$

with  $a_\alpha$  and  $a_0$  being given mappings, then it is called a **quasilinear** equation.

A PDE is called **fully nonlinear** if the highest order derivative is nonlinear.

**Theorem 1.1.** (*Superposition principle for homogeneous linear equations*). Let  $f \equiv 0$  in (1.3). If  $u_1$  and  $u_2$  are both solutions of (1.3), then all the linear combinations of  $u_1$  and  $u_2$  are still solutions of (1.3).

We list here some examples of partial differential equations.

**Linear PDEs**

- (1) Linear transport equation  $u_t + \mathbf{b} \cdot \nabla u = f$ ,
- (2) Wave equation  $u_{tt} - \Delta u = f$ ,
- (3) Heat equation  $u_t - \Delta u = f$ ,
- (4) Poisson equation  $-\Delta u = f$ ,
- (5) Schrödinger equation  $iu_t = -\Delta u$ ,
- (6) .....

where we use the notation  $\nabla = (\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n})$  for  $n$ -dimensional gradient operator and  $\Delta = \nabla \cdot \nabla = \partial_{x_1x_1} + \partial_{x_2x_2} + \dots + \partial_{x_nx_n}$  the  $n$ -dimensional Laplacian.

There are also some other higher order linear equations, which are mostly from physics (see more examples in Evan’s book).

**Nonlinear PDEs** There are plenty of nonlinear equations in the literature. But we can barely include them in this course. However, if time allows, we will give a short introduction on the 1-D scalar first order hyperbolic conservation laws.

- (1) Scalar conservation law  $u_t + \operatorname{div} \mathbf{F}(u) = 0$ ,
- (2) Hamilton-Jacobi equation  $u_t + H(Du, x) = 0$ ,
- (3) Nonlinear Poisson equation  $-\Delta u = f(u)$ ,
- (4) Monge-Ampère equation  $\det D^2u = f$ ,
- (5) Porous medium equation  $u_t - \Delta u^m = 0$ ,
- (6) Nonlinear Schrödinger equation  $iu_t = -\Delta u + |u|^2u$ ,
- (7) .....

PDE systems are as well important in the literature, such as Euler system, Navier-Stokes system, Maxwell’s equations, reaction diffusion system, multi-component diffusion system, Keller-Segel system. However, due to the limitation of time, we are not going to investigate any system in this course.

**1.2. Setup of the problems.** In order to set up a well-posed PDE problem, additional boundary and initial conditions need to be given.

For time evolutionary PDE, some initial data of the problem is desired. For example, for heat equation  $u_t - \Delta u = f$ , we need initial conditions like  $u|_{t=0} = u_0(x)$  where  $u_0(x)$ , is a given function. For wave equation  $u_{tt} - \Delta u = f$ , we need  $u|_{t=0} = g$  and  $u_t|_{t=0} = h$  since we have double derivatives in time  $t$ .

If we study the PDE with space variable  $x$  in  $\Omega$ , which is an open subset in  $\mathbb{R}^n$ , and  $\partial\Omega \neq \emptyset$ , we need to give boundary conditions. There are three kinds of typical boundary conditions, each of which has its own physical background.

- (1) **Dirichlet boundary condition**, the unknown itself is given on the boundary

$$u|_{\partial\Omega} = u_D(x),$$

where  $u_D(x)$  is a given function defined on  $\partial\Omega$ .

- (2) **Neumann boundary condition**, the normal derivative of the unknown is given on the boundary

$$\nabla u \cdot \gamma|_{\partial\Omega} = u_N(x),$$

where  $\gamma$  is the pointwise unit outer normal vector of  $\partial\Omega$ .

- (3) **Robin boundary condition**, the nontrivial linear combination of the unknown and its normal derivative is given on the boundary

$$\alpha u + \beta \nabla u \cdot \gamma|_{\partial\Omega} = u_R(x),$$

where  $\alpha(x)$  and  $\beta(x)$  are nonnegative functions given on the boundary.

**1.3. Basic topics in studying PDE problems.** After settling down a reasonable PDE problem, our main purpose is to find the solution and to study it in order to have a more clear understanding of the solution behavior. A direct way is to find an analytical solution representation, which gives explicitly how the solution behaves. However, in most of the cases, it is impossible to find an explicit formula for the solution. What can we do then? In working with the problems directly, we can still get detailed information of the solutions without knowing the explicit representation.

Given a PDE problem, we need to study its well-posedness, including

- (1) **Existence.** As mentioned before that the direct way to get existence of solution is to solve it explicitly. Then under appropriate regularity assumption of the given data, one can show that the existence of classical solution is given by solution formula. Here classical solution means that the  $k$ -th order derivatives of the solution exist if it is a solution to a  $k$ -th order PDE. For those equations, for which it is hopeless to get solution formula, there are several other ways to get existence, for example, fixed point theorems from functional analysis, variational methods, approximation by approximated solutions are oftenly used in the literature. The main thing one should keep in mind is, in which function class one could expect the existence of solution. Many problems have no classical solution. Usually the existence should be studied in the sense that the equation is satisfied in some weaker formulation instead of its classical one. Afterwards, one can try to find out whether the weak solution has further regularities if the given data are correspondingly smooth.
- (2) **Uniqueness.** Once we have existence at hand, a natural question to ask is whether it is unique and of course in which class it is unique. For some of the problems one may also expect to get existence of multi solutions. The larger the existence class, the less the hope to assure uniqueness.
- (3) **Stability.** The stability of a solution means that if the given data are slightly perturbed, is it true that the solution also doesn't change so much? In other words, stability means the continuous dependence of the solution on its given data, including initial and boundary conditions, or any other given parameters in the equation.
- (4) **Solution behavior.** From the aspect of applications, solution behavior is the most important thing that the PDE analysis should contribute to. Here the solution

behavior we talk about is in the very general sense. It could be the large time behavior, boundary layer behavior or singular behavior if one considers the singular limit problem.

In the literature, the research in PDEs starts from finding explicit solutions. The basic tools that we will discuss in this course are

- (1) method of characteristics,
- (2) Fourier transform,
- (3) separation of variables (Fourier series),
- (4) Green's function.

Although these tools to find solution formula couldn't be directly used in many of the PDE problems, they still play a very important role in understanding the basic theories and phenomena.

There are also many other tools to study the solution behavior without using formula, to name just a few,

- (1) energy method,
- (2) maximum principle,
- (3) asymptotic expansion,
- (4) . . . . .

As explained above, for many of the physical problems, one couldn't expect that they always have classical solutions. In this course, we will also talk about the ideas on how to define weak solutions. Moreover, a very brief introduction on distributions will be given.

**1.4. Linear transport equations.** In the end of introduction, we use linear transport equation as an example to give a first try to get solution representation. It is actually an application of the solution for first order ordinary differential equation.

1.4.1. *Constant speed.* Linear transport equation in 1-D is the simplest partial differential equation,

$$\begin{aligned}\rho_t + a\rho_x &= 0, & \text{in } \mathbb{R} \times \mathbb{R}_+, \\ \rho(x, 0) &= \rho_0(x),\end{aligned}$$

where  $a$  is a constant. This equation with constant speed is an ordinary differential system in the sense that

$$\begin{aligned}\frac{d}{dt}\rho(x(t), t) &= 0, \\ \frac{d}{dt}x(t) &= a, \\ \rho(x(0), 0) &= \rho_0(x_0).\end{aligned}$$

Obviously, it has solution

$$\rho(x, t) = \rho_0(x - at).$$

This strategy is called the **method of characteristics** and  $x(t)$  is called the characteristic line.

Moreover, one can also solve the transport equation in inhomogeneous case

$$\begin{aligned}\rho_t + a\rho_x &= f(x, t), & \text{in } \mathbb{R} \times \mathbb{R}_+, \\ \rho(x, 0) &= \rho_0(x)\end{aligned}$$

by using the same characteristic line  $x(t) = x_0 + at$ . The solution is

$$\rho(x, t) = \rho_0(x - at) + \int_0^t f(x - a(t - s), s) ds.$$

*Remark 1.1.* By using the method of characteristics, one can easily solve the multi-D equation

$$\begin{aligned}\rho_t + \mathbf{b} \cdot \nabla \rho &= 0, & \text{in } \mathbb{R}^n \times \mathbb{R}_+, \\ \rho(x, 0) &= \rho_0(x),\end{aligned}$$

where  $\mathbf{b}$  is a constant vector. We leave it as an exercise.

1.4.2. *Non-constant speed.* The Cauchy problem we consider in this part is

$$\begin{aligned}\rho_t + (v(x)\rho)_x &= 0, & \text{in } \mathbb{R} \times \mathbb{R}_+, \\ \rho(x, 0) &= \rho_0(x),\end{aligned}\tag{1.6}$$

where  $v(x)$  is a given Lipschitz continuous function.

A reformulation of the equation is

$$\rho_t + v(x)\rho_x + v'(x)\rho = 0.$$

If  $v(x)$  is Lipschitz continuous, then the characteristic line  $x(t)$  satisfies

$$\begin{aligned}\frac{dx}{dt} &= v(x), \\ x(0) &= x_0.\end{aligned}\tag{1.7}$$

With the help of this line, the equation is rewritten into

$$\begin{aligned}\frac{d\rho(x(t; x_0), t)}{dt} &= -v'(x(t; x_0))\rho, \\ \rho(x(t; x_0), t)|_{t=0} &= \rho_0(x_0).\end{aligned}$$

By separation of variables in solving ODE, we have

$$\begin{aligned}\ln \rho(x(t; x_0), t) &= \ln \rho_0(x_0) + \int_0^t -v'(x(\tau; x_0)) d\tau \\ &= \ln \rho_0(x_0) + \int_{x_0}^{x(t; x_0)} -\frac{v'(x)}{v(x)} dx \\ &= \ln \rho_0(x_0) - \ln v(x(t; x_0)) + \ln v(x_0),\end{aligned}$$

where in the first equation the change of variable  $x = x(\tau, x_0)$ ,  $dx = x'd\tau = v(x)d\tau$  is used. So the problem has solution

$$\rho(x(t; x_0), t) = \rho_0(x_0) \frac{v(x_0)}{v(x(t; x_0))}.$$

Therefore, finding the exact solution of the homogeneous problem (1.6) is reduced to the solvability of the characteristic line (1.7).

**1.5. Half-line problem.** We will study the half-line problem for transport equation only with constant speed. One must be careful in giving boundary conditions because of the “directions” of the characteristic lines. One can not arbitrarily give boundary condition at  $x = 0$  if the characteristic lines starting from  $t = 0$  meet  $x = 0$  at some time  $t$ . As an example, we consider

$$\begin{aligned} u_t + u_x &= 0, & (x, t) &\in (0, +\infty) \times (0, +\infty), \\ u|_{t=0} &= u_0(x), & u|_{x=0} &= 0. \end{aligned}$$

For compatibility, we need  $u_0(0) = 0$ . It is easily seen that the solution is

$$u(x, t) = \begin{cases} 0, & x \leq t, \\ u_0(x - t), & x > t. \end{cases}$$

If the boundary condition is inhomogeneous, say  $u|_{x=0} = g(t)$ , the compatibility condition is then  $u(0, 0) = g(0)$ . The solution changes accordingly to

$$u(x, t) = \begin{cases} g(t - x), & x \leq t, \\ u_0(x - t), & x > t. \end{cases}$$

Another method is to use the transformation  $v(x, t) = u(x, t) - g(t)$ , so that  $v$  is zero on the boundary  $x = 0$ . And the problem that  $v$  satisfies is

$$\begin{aligned} v_t + v_x &= -g_t(t), & (x, t) &\in (0, +\infty) \times (0, +\infty), \\ v|_{t=0} &= u_0(x) - g(0), & v|_{x=0} &= 0. \end{aligned}$$

**1.6. Problems.**

- (1) Point out the type of these equations (linear, semilinear, quasilinear, fully nonlinear).
  - (a)  $u_t - u_x u_{xxx} = x^2$ ,
  - (b)  $-\Delta u + u^2 = 1$ ,
  - (c)  $u_{tt} - \operatorname{div}((x^2 + t)\nabla u) = f(x)$ ,
  - (d)  $u_t + \operatorname{div}\left[u\nabla\left(\frac{\Delta\sqrt{u}}{\sqrt{u}}\right)\right] = 0$ ,
  - (e)  $\Delta(u^2) = f(x)$ ,
  - (f)  $|\nabla u| = 1$ .
- (2) Find the solution formula by using characteristic method.
  - (a)  $\begin{cases} u_t + (1 + x^2)u_x - u = 0, & t > 0, -\infty < x < \infty, \\ u|_{t=0} = \arctan x, & -\infty < x < \infty, \end{cases}$

$$(b) \begin{cases} u_t + \mathbf{b} \cdot \nabla u = f, & (x, t) \in \mathbb{R}^n \times (0, \infty), \\ u|_{t=0} = g, & x \in \mathbb{R}^n. \end{cases}$$

## 2. WAVE EQUATION

After a short derivation of the one dimensional wave equation, we will give the solution formula for initial value problem (the Cauchy problem) by using characteristic method. Then with the help of spherical mean, we will derive the three dimensional Kirchhoff's formula and two dimensional Poisson's formula by Hadamard's method of descent. These shows that if the initial data are regular enough, then the classical solutions exist. Furthermore, the uniqueness and stability results will be given by using energy method.

For one dimensional initial boundary value problem with Dirichlet boundary condition, we will use the separation of variable to deduce a solution formula with the help of Fourier series. In the mean while, the Sturm Liouville theorem is going to be presented which provides the theoretical basis for the method of separation of variables. Based on that, the existence of solution can be obtained by studying the convergence of function series. Again energy method will be used to show uniqueness and stability.

In the end of this section, we introduce the weak solution of initial boundary value problem in the sense of distribution and show that the weak solution exists uniquely.

**2.1. Derivation of the one dimensional wave equation.** A stretched string with constant density  $\rho$  is set to vibrate on a plane. The displacement of the string  $u(x, t)$  is a function of  $x$  and time  $t$ , where  $x$  is the horizontal coordinate. Now let's consider a discretized model in which a sequence of equally massed (i.e.,  $m = \rho h$ ) particles with equal distance  $h$  connect each other with coordinates  $\dots, x_{n-1}, x_n, x_{n+1}, \dots$ . Newton's second law implies that for the  $n$ -th particle, the external force on this particle is equal to its mass multiplied by the acceleration of it. We further make a simple assumption that the external force  $F$  on the  $n$ -th particle, which comes from the  $(n - 1)$ -th and the  $(n + 1)$ -th particle, is proportional to the difference of the displacements of its adjacent particles and itself, i.e.

$$F = (u(x_{n+1}, t) - u(x_n, t))/h + (u(x_{n-1}, t) - u(x_n, t))/h.$$

Therefore

$$\rho h u''(x_n, t) = \frac{u(x_{n+1}, t) + u(x_{n-1}, t) - 2u(x_n, t)}{h}.$$

Now if we take the limit  $h \rightarrow 0$  and come back to the continuous case, we arrive at the one dimensional wave equation

$$\rho u_{tt} = u_{xx}.$$

## 2.2. Cauchy problem.

**2.2.1. Solution formula and existence. d'Alembert's formula — 1-D** We will give the formal solution of Cauchy problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0, & \text{in } \mathbb{R} \times \mathbb{R}_+, \\ u|_{t=0} &= g(x), \\ u_t|_{t=0} &= h(x). \end{aligned} \tag{2.1}$$

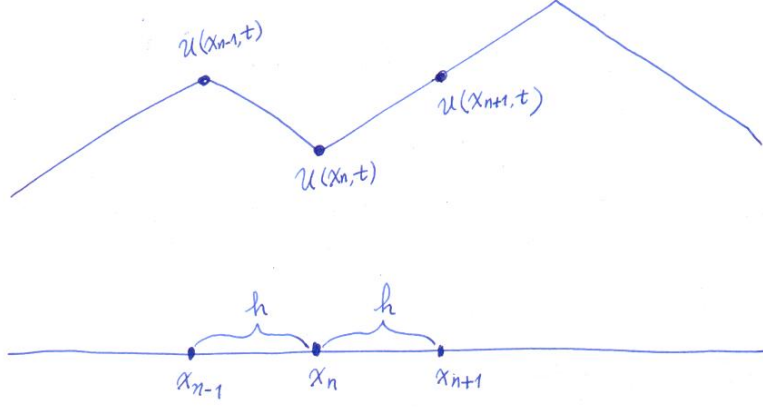


FIGURE 1. Derivation of wave equation by particle method

By factorizing the operator  $\partial_{tt} - \partial_{xx} = (\partial_t + \partial_x)(\partial_t - \partial_x)$ , the problem is reduced into solving the following two transport equations,

$$\begin{aligned} v_t + v_x &= 0, \\ v|_{t=0} &= h(x) - g'(x) \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} u_t - u_x &= v, \\ u|_{t=0} &= g(x). \end{aligned} \quad (2.3)$$

With the help of characteristic method, the solution of (2.2) is

$$v(x, t) = h(x - t) - g'(x - t)$$

and the solution of (2.3) is

$$u(x, t) = g(x + t) + \int_0^t v(x + (t - s), s) ds.$$

Therefore, combining the two formulas above, we have the solution of (2.1),

$$\begin{aligned} u(x, t) &= g(x + t) + \int_0^t h(x + t - 2s) - g'(x + t - 2s) ds \\ &= g(x + t) + \frac{1}{2} \int_{x-t}^{x+t} (h(y) - g'(y)) dy \\ &= g(x + t) - \frac{1}{2} g(x + t) + \frac{1}{2} g(x - t) + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy \\ &= \frac{1}{2} (g(x + t) + g(x - t)) + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy. \end{aligned}$$

This is the so-called **d'Alembert's formula**. The formal solution of (2.1) is

$$u(x, t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy. \quad (2.4)$$

From this formula, we are ready to get the existence of the solution with smooth initial data

**Theorem 2.1.** *If  $g \in C^2(\mathbb{R})$  and  $h \in C^1(\mathbb{R})$ , then  $u$ , given by d'Alembert's formula, is a  $C^2(\mathbb{R} \times [0, +\infty))$ -function. Furthermore, it satisfies wave equation  $u_{tt} - u_{xx} = 0$  and*

$$\lim_{(x,t) \rightarrow (x_0,0)} u(x, t) = g(x_0), \quad \lim_{(x,t) \rightarrow (x_0,0)} u_t(x, t) = h(x_0).$$

**Some properties of the solution** d'Alembert's formula reveals in itself some important sets in the  $x-t$  space.

- (1)  $\{y \in \mathbb{R} \mid |y-x| \leq t\}$  is called the **domain of dependence** of the point  $(x, t)$ ,
- (2)  $\{(x, t) \in \mathbb{R} \times [0, +\infty) \mid x \geq x_1 - t \text{ and } x \leq x_2 + t\}$  is called the **range of influence** of  $[x_1, x_2]$ ,
- (3)  $\{(x, t) \in \mathbb{R} \times [0, +\infty) \mid x \geq x_1 + t \text{ and } x \leq x_2 - t\}$  is called the **determining region** of  $[x_1, x_2]$ ,
- (4)  $x+t$  and  $x-t$  are the **characteristics** of the wave equation.

**Inhomogeneous problem** By the same method, we can also find the solution of inhomogeneous problem

$$\begin{aligned} u_{tt} - u_{xx} &= f(x, t), & (x, t) &\in \mathbb{R} \times \mathbb{R}_+, \\ u|_{t=0} &= g(x), & u_t|_{t=0} &= h(x), \end{aligned} \quad (2.5)$$

which is

$$u(x, t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy + \frac{1}{2} \int_0^t ds \int_{x-(t-s)}^{x+(t-s)} f(y, s) dy. \quad (2.6)$$

By using the solution formula, it is easy to verify that

**Corollary 2.1.** *If  $g$ ,  $h$  and  $f$  are odd (even, or periodic) in  $x$ , so is  $u$ .*

**Half-line problem** By using extension, we can also give the solution formula of the following problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0, & \text{in } \mathbb{R}_+ \times \mathbb{R}_+, \\ u|_{t=0} &= g(x), & u_t|_{t=0} &= h(x), \\ u|_{x=0} &= 0. \end{aligned} \quad (2.7)$$

In order to assure compatibility between initial and boundary conditions, we need  $h(0) = g(0) = 0$ . The homogeneous Dirichlet boundary condition at  $x = 0$  motivates us to use odd extension. Let

$$\tilde{g} = \begin{cases} g(x), & x \geq 0, \\ -g(-x), & x < 0, \end{cases}$$

and we do the same odd extensions for  $\tilde{u}(x, t)$  and  $\tilde{h}(x)$ . Then from the above corollary,  $\tilde{u}$  satisfies the following Cauchy problem

$$\begin{aligned}\tilde{u}_{tt} - \tilde{u}_{xx} &= 0, & \text{in } \mathbb{R} \times \mathbb{R}_+, \\ \tilde{u}|_{t=0} &= \tilde{g}(x), \\ \tilde{u}_t|_{t=0} &= \tilde{h}(x).\end{aligned}$$

By d'Alembert's formula,  $\tilde{u}$  has the representation

$$\tilde{u}(x, t) = \frac{1}{2}(\tilde{g}(x+t) + \tilde{g}(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} \tilde{h}(y) dy.$$

We need to get back to the domain  $\{(x, t) : x > 0, t > 0\}$  and drop the tildes in the formula. There are two cases. In the case of  $x \geq t$ , our solution has the representation

$$u(x, t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy.$$

In the case of  $0 \leq x < t$ , the solution is given by

$$\begin{aligned}u(x, t) &= \frac{1}{2}(g(x+t) - g(t-x)) + \frac{1}{2} \left( \int_0^{x+t} h(y) dy - \int_0^{t-x} h(-y) dy \right) \\ &= \frac{1}{2}(g(x+t) - g(t-x)) + \frac{1}{2} \int_{t-x}^{x+t} h(y) dy.\end{aligned}$$

*Remark 2.1.* For inhomogeneous boundary condition  $u|_{x=0} = u_D(t)$ , we can first get a homogeneous boundary condition by using new variable  $v = u - u_D(t)$ , where  $v$  satisfies  $v_{tt} - v_{xx} = -(u_D)_{tt}$ . Afterwards, we do the same odd extension to get the solution formula.

*Remark 2.2.* It is an easy exercise to get half-line problem with homogeneous Neumann boundary condition  $u_x|_{x=0} = 0$  by using even extension.

**Kirchhoff's formula in 3-D and Poisson's formula in 2-D** We will reduce the multi-dimension problem into a half-line problem by using **spherical mean** of the solution.

The spherical mean of a function  $u(x, t)$  on  $\partial B(x, r)$  is given by

$$U(x; r, t) = \int_{\partial B(x, r)} u(y, t) dS_y. \quad (2.8)$$

**Lemma 2.1.** *If  $u \in C^m(\mathbb{R}^n \times (0, +\infty))$  is a solution of*

$$\begin{aligned}u_{tt} - \Delta u &= 0, & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u|_{t=0} &= g, & u_t|_{t=0} = h.\end{aligned} \quad (2.9)$$

*Then the spherical mean of  $u$ ,  $U(x; r, t)$  as a function of  $r$  and  $t$  is a function in  $C^m([0, +\infty) \times [0, +\infty))$ . Furthermore it satisfies*

$$\begin{aligned}U_{tt} - U_{rr} - \frac{n-1}{r} U_r &= 0, & \text{in } \mathbb{R}_+ \times \mathbb{R}_+, \\ U|_{t=0} &= G, & U_t|_{t=0} = H,\end{aligned} \quad (2.10)$$

where  $G, H$  are the corresponding spherical mean of  $g, h$ . The equation (2.10) is called the **Euler-Poisson-Darboux equation**.

*Proof.* By direct calculations, we have

$$\begin{aligned} U_r(x; r, t) &= \frac{\partial}{\partial r} \int_{\partial B(x,r)} u(y, t) dS_y = \frac{\partial}{\partial r} \int_{\partial B(0,1)} u(x + rz, t) dS_z \\ &= \int_{\partial B(0,1)} \nabla u(x + rz, t) \cdot z dS_z = \int_{\partial B(x,r)} \nabla u(y, t) \cdot \frac{y-x}{r} dS_y \\ &= \int_{\partial B(x,r)} \nabla u \cdot \gamma dS_y = \frac{r}{n} \int_{B(x,r)} \Delta u(y, t) dy. \end{aligned}$$

As a consequence,

$$\lim_{r \rightarrow 0^+} U_r(x; r, t) = 0.$$

If we take one derivative more,

$$\begin{aligned} U_{rr}(x; r, t) &= \frac{\partial}{\partial r} \left( \frac{r}{n} \int_{B(x,r)} \Delta u(y, t) dy \right) = \frac{1}{n\alpha(n)} \frac{\partial}{\partial r} \left( r^{1-n} \int_{B(x,r)} \Delta u(y, t) dy \right) \\ &= \frac{1-n}{n} \frac{1}{\alpha(n)r^n} \int_{B(x,r)} \Delta u(y, t) dy + \frac{1}{n\alpha(n)r^{n-1}} \frac{\partial}{\partial r} \int_{B(x,r)} \Delta u(y, t) dy \\ &= \left( \frac{1}{n} - 1 \right) \int_{B(x,r)} \Delta u dy + \int_{\partial B(x,r)} \Delta u dS_y. \end{aligned}$$

where  $\alpha(n) = \pi^{\frac{n}{2}} / \Gamma(\frac{n}{2} + 1)$  is the volume of the  $n$  dimensional unit ball. Furthermore,

$$\lim_{r \rightarrow 0^+} U_{rr}(x; r, t) = \frac{1}{n} \Delta u(x, t).$$

Therefore, if  $u \in C^2$ , we have  $U \in C^2$ . By iteration argument, we have  $U \in C^m$  if  $u \in C^m$ .

Back to the first order derivative, by using the wave equation  $u_{tt} - \Delta u = 0$ , we have

$$U_r = \frac{r}{n} \int_{B(x,r)} u_{tt} dy = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} u_{tt} dy.$$

Multiplying the above equation by  $r^{n-1}$  gives

$$r^{n-1} U_r = \frac{1}{n\alpha(n)} \int_{B(x,r)} u_{tt} dy.$$

Then the desired equation follows from taking one more derivative of  $r^{n-1} U_r$ , i.e.,

$$(r^{n-1} U_r)_r = \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt} dS_y = r^{n-1} \int_{\partial B(x,r)} u_{tt} dS_y = r^{n-1} U_{tt}.$$

□

In the case of  $n = 3$ , we will get Kirchhoff's formula by using Euler-Poisson-Darboux equation.

Let  $\tilde{U} = rU$ ,  $\tilde{G} = rG$  and  $\tilde{H} = rH$ , we have

$$\tilde{U}_r = U + rU_r,$$

and moreover

$$\tilde{U}_{tt} = rU_{tt} = rU_{rr} + 2U_r = (U + rU_r)_r = \tilde{U}_{rr}.$$

Now  $\tilde{U}$  solves the half-line problem

$$\begin{aligned} \tilde{U}_{tt} - \tilde{U}_{rr} &= 0, & \text{in } \mathbb{R}_+ \times \mathbb{R}_+, \\ \tilde{U}|_{t=0} &= \tilde{G}, & \tilde{U}_t|_{t=0} = \tilde{H}, \\ \tilde{U}|_{r=0} &= 0. \end{aligned}$$

By the solution representation in half-line problem, we have

$$\tilde{U}(x; r, t) = \frac{1}{2}(\tilde{G}(r+t) - \tilde{G}(t-r)) + \frac{1}{2} \int_{t-r}^{r+t} \tilde{H}(y) dy, \quad \forall 0 < r < t.$$

Since  $u(x, t)$  is a continuous function, its value at  $(x, t)$  is exactly the limit of its spherical mean. Thus

$$\begin{aligned} u(x, t) &= \lim_{r \rightarrow 0^+} U(x; r, t) = \lim_{r \rightarrow 0^+} \frac{\tilde{U}(x; r, t)}{r} \\ &= \lim_{r \rightarrow 0^+} \left[ \frac{1}{2} \frac{\tilde{G}(r+t) - \tilde{G}(t-r)}{r} + \frac{1}{2r} \int_{t-r}^{r+t} \tilde{H}(y) dy \right] \\ &= \tilde{G}'(t) + \tilde{H}(t). \end{aligned}$$

By the definition of  $\tilde{G}$  and  $\tilde{H}$ , after changing back to the original functions  $g$  and  $h$ , we arrive at

$$u(x, t) = \frac{\partial}{\partial t} \left[ t \int_{\partial B(x, t)} g(y) dS_y \right] + t \int_{\partial B(x, t)} h(y) dS_y.$$

Therefore after the following further computation

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\partial B(x, t)} g(y) dS_y &= \frac{\partial}{\partial t} \int_{\partial B(0, 1)} g(x + tz) dS_z \\ &= \int_{\partial B(0, 1)} \nabla g(x + tz) \cdot z dS_z = \int_{\partial B(x, t)} \nabla g(y) \cdot \frac{y - x}{t} dS_y, \end{aligned}$$

we obtain the 3-D **Kirchhoff's formula**,

$$u(x, t) = \int_{\partial B(x, t)} [g(y) + \nabla g(y) \cdot (y - x) + th(y)] dS_y. \quad (2.11)$$

In the case of  $n = 2$ , we will get Poisson's formula by the Hadamard's method of descent.

If  $u(x_1, x_2, t)$  is a solution in 2-D, let  $\bar{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t)$ , then  $\bar{u}$  solves the wave equation in 3-D,

$$\begin{aligned} \bar{u}_{tt} - \Delta \bar{u} &= 0, & \text{in } \mathbb{R}^3 \times (0, \infty), \\ \bar{u}|_{t=0} &= \bar{g}, & \bar{u}_t|_{t=0} = \bar{h}, \end{aligned}$$

where  $\bar{g}(x_1, x_2, x_3) = g(x_1, x_2)$  and  $\bar{h}(x_1, x_2, x_3) = h(x_1, x_2)$ . Then by Kirchhoff's formula in 3-D, we have

$$u(x, t) = \frac{\partial}{\partial t} \left[ t \int_{\partial B(\bar{x}, t)} \bar{g}(y) dS_y \right] + t \int_{\partial B(\bar{x}, t)} \bar{h}(y) dS_y,$$

where  $\bar{x} = (x_1, x_2, 0)$ . Due to the fact that  $\bar{g}(y_1, y_2, y_3) = g(y_1, y_2)$ , we can simplify the integral on  $\partial B(\bar{x}, t)$  by

$$\begin{aligned} \int_{\partial B(\bar{x}, t)} \bar{g}(y) dS_y &= \frac{1}{4\pi t^2} \int_{\partial B(\bar{x}, t)} \bar{g}(y) dS_y \\ &= \frac{2}{4\pi t^2} \int_{B(x, t)} g(y) (1 + |\nabla \gamma(y)|^2)^{\frac{1}{2}} dy, \end{aligned}$$

where  $\gamma(y) = \sqrt{t^2 - (y - x)^2}$  and  $(1 + |\nabla \gamma(y)|^2)^{\frac{1}{2}} = t(t^2 - |y - x|^2)^{-\frac{1}{2}}$ . Therefore

$$\begin{aligned} \int_{\partial B(\bar{x}, t)} \bar{g}(y) dS_y &= \frac{1}{2\pi t} \int_{B(x, t)} \frac{g(y)}{(t^2 - |y - x|^2)^{\frac{1}{2}}} dy \\ &= \frac{t}{2} \int_{B(x, t)} \frac{g(y)}{(t^2 - |y - x|^2)^{\frac{1}{2}}} dy. \end{aligned}$$

Then after taking derivative with respect to  $t$ , we have

$$\begin{aligned} &\frac{\partial}{\partial t} \left[ t^2 \int_{B(x, t)} \frac{g(y)}{(t^2 - |y - x|^2)^{\frac{1}{2}}} dy \right] = \frac{\partial}{\partial t} \left[ t \int_{B(0, 1)} \frac{g(x + tz)}{(1 - |z|^2)^{\frac{1}{2}}} dz \right] \\ &= \int_{B(0, 1)} \frac{g(x + tz)}{(1 - |z|^2)^{\frac{1}{2}}} dz + t \int_{B(0, 1)} \frac{\nabla g(x + tz) \cdot z}{(1 - |z|^2)^{\frac{1}{2}}} dz \\ &= t \int_{B(x, t)} \frac{g(y)}{(t^2 - |y - x|^2)^{\frac{1}{2}}} dy + t \int_{B(x, t)} \frac{\nabla g(y) \cdot (y - x)}{(t^2 - |y - x|^2)^{\frac{1}{2}}} dy \end{aligned}$$

Thus the 2-D **Poisson's formula** is

$$u(x, t) = \frac{1}{2} \int_{B(x, t)} \frac{tg(y) + t^2 h(y) + t \nabla g(y) \cdot (y - x)}{(t^2 - |y - x|^2)^{\frac{1}{2}}} dy. \quad (2.12)$$

With the help of Kirchhoff's and Poisson's formula in 3-D and 2-D, we have the following existence result.

**Theorem 2.2.** ( $n = 2, 3$ ) If  $g \in C^3(\mathbb{R}^n)$  and  $h \in C^2(\mathbb{R}^n)$ , then  $u$  given by (2.11) and (2.12), a function in  $C^2(\mathbb{R}^n \times [0, +\infty))$ , is a classical solution of (2.9).

**The difference of the solution behaviors between 3-D and 2-D** If we look at the Kirchhoff's formula (2.11) and the Poisson's formula (2.12), we can easily find that the main difference locates on the domain of the integrals. The integral is on the sphere (which is the boundary of a domain) in Kirchhoff's formula, while the integral is in the ball in Poisson's formula.

Let's assume that the initial data have compact support  $\Omega$ , where  $\Omega$  is connected and regular enough. For any fixed point  $x_0 \notin \Omega$ , let  $d_1 = \text{dist}(x_0, \Omega) > 0$ ,  $d_2 = \max\{\text{dist}(x_0, x) : x \in \Omega\}$ . In 3-D case, the possible non-zero points of  $u(x_0, t)$  can only be in the interval  $t \in [d_1, d_2]$ , while in 2-D, the possible non-zero points of  $u(x_0, t)$  must be the half line  $t \in [d_1, +\infty)$ . That explains why one can hear the others' voice in 3-D, and why the water wave diffuses to the whole space in 2-D.

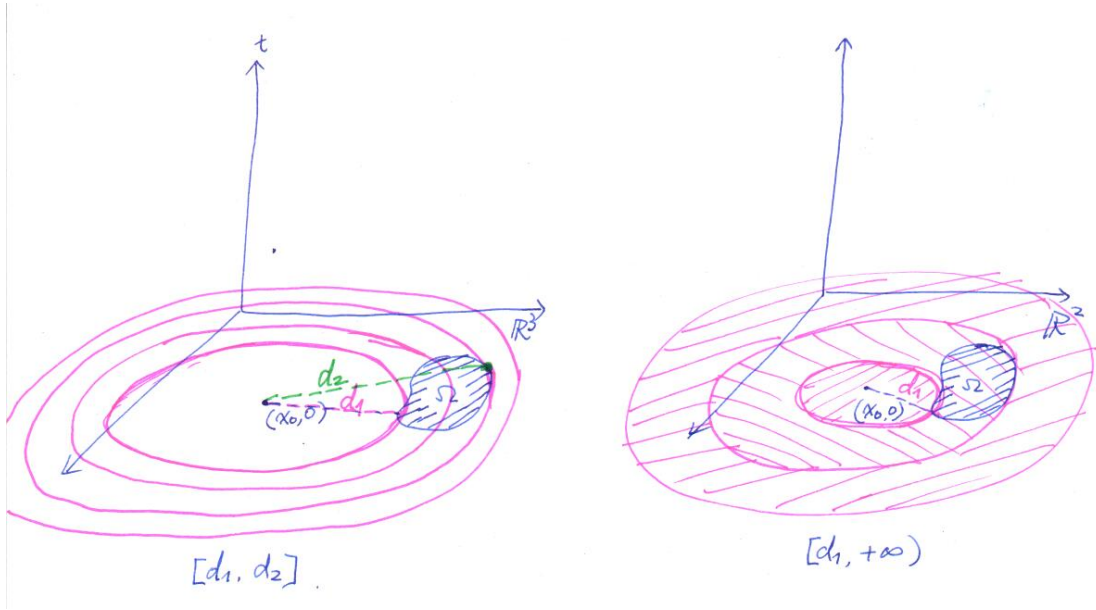


FIGURE 2. Wave propagation in 3-D and 2-D

2.2.2. *Uniqueness and stability (Energy method).* After obtaining the existence result, a further natural question to ask is whether the solution is unique and stable? In the following discussion, we introduce the energy method, which is a “powerful” tool in PDE theory. Uniqueness and stability follow thereafter as an application of the energy estimates.

We first introduce a useful lemma.

**Lemma 2.2.** (*Gronwall's inequality*) Assume  $G(\tau) \geq 0$ ,  $G'(\tau) \in C[0, T]$  and  $\forall \tau \in [0, T]$ , the following inequality holds

$$\frac{dG(\tau)}{d\tau} \leq CG(\tau) + F(\tau),$$

where  $C$  is a nonnegative constant,  $F(\tau) \geq 0$  is nondecreasing in  $\tau$ . Then

$$\frac{dG(\tau)}{d\tau} \leq Ce^{C\tau}G(0) + e^{C\tau}F(\tau),$$

and

$$G(\tau) \leq e^{C\tau}G(0) + C^{-1}(e^{C\tau} - 1)F(\tau).$$

*Proof.* By multiplying the given inequality by  $e^{-C\tau}$  and integrating it over  $[0, \tau]$ , we have

$$e^{-C\tau}G(\tau) \leq G(0) + \int_0^\tau e^{-Ct}F(t)dt \leq G(0) + F(\tau)C^{-1}(1 - e^{-C\tau}).$$

□

The Cauchy problem we considered is revisited

$$\begin{aligned} u_{tt} - u_{xx} &= f, & \text{in } \mathbb{R} \times \mathbb{R}_+, \\ u|_{t=0} &= g(x), \\ u_t|_{t=0} &= h(x). \end{aligned} \tag{2.13}$$

The energy inequality of 1-D Cauchy problem (2.13) is

**Theorem 2.3.** *If  $u \in C^1(\mathbb{R} \times [0, +\infty)) \cap C^2(\mathbb{R} \times (0, +\infty))$  is a solution of (2.13), then  $\forall (x_0, t_0) \in \mathbb{R} \times (0, +\infty)$ , we have*

$$\begin{aligned} \int_{\Omega_\tau} [u_t^2(x, \tau) + u_x^2(x, \tau)] dx &\leq C(t_0) \left( \int_{\Omega_0} (h^2 + g_x^2) dx + \int \int_{K_\tau} f^2(x, t) dx dt \right), \\ \int \int_{K_\tau} [u_t^2(x, t) + u_x^2(x, t)] dx dt &\leq C(t_0) \left( \int_{\Omega_0} (h^2 + g_x^2) dx + \int \int_{K_\tau} f^2(x, t) dx dt \right), \end{aligned}$$

where  $K = \{(x, t) \in \mathbb{R} \times [0, +\infty) : |x - x_0| < t_0 - t\}$ ,  $K_\tau = K \cap \{0 \leq t \leq \tau\}$ ,  $\Omega_\tau = \bar{K} \cap \{t = \tau\}$ , and  $C(t_0)$  is a constant depends on  $t_0$ .

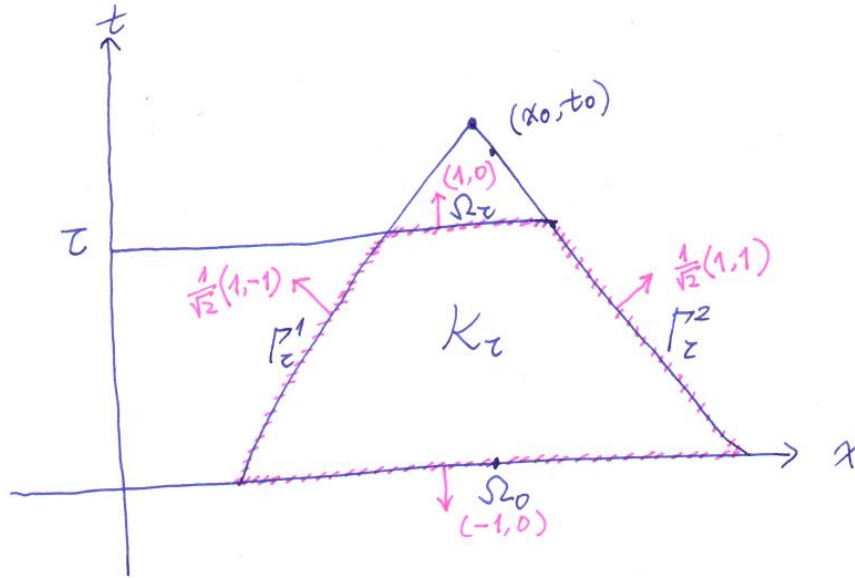


FIGURE 3. Domains in energy estimates

*Proof.* Multiplying the equation by  $u_t$  and integrating over  $K_\tau$ , we have

$$\int \int_{K_\tau} (u_t u_{tt} - u_t u_{xx}) dx dt = \int \int_{K_\tau} u_t f dx dt.$$

Noticing that the boundary of  $K_\tau$  is  $\partial K_\tau = \Omega_0 \cup \Omega_\tau \cup \Gamma_\tau^1 \cup \Gamma_\tau^2$ , we can calculate the left hand side by using divergence lemma,

$$\begin{aligned}
& \int \int_{K_\tau} \frac{1}{2}(u_t^2 + u_x^2)_t dxdt - \int \int_{K_\tau} (u_t u_x)_x dxdt \\
&= \int_{\partial K_\tau} \left( \frac{1}{2}(u_t^2 + u_x^2), -u_t u_x \right)^T \cdot \gamma dl \\
&= \int_{\Omega_\tau} \frac{1}{2}(u_t^2 + u_x^2) dx - \int_{\Omega_0} \frac{1}{2}(u_t^2 + u_x^2) dx \\
&\quad + \int_{\Gamma_\tau^1} \frac{1}{\sqrt{2}} \left( \frac{1}{2}(u_t^2 + u_x^2) + u_t u_x \right) dl + \int_{\Gamma_\tau^2} \frac{1}{\sqrt{2}} \left( \frac{1}{2}(u_t^2 + u_x^2) - u_t u_x \right) dl \\
&\geq \int_{\Omega_\tau} \frac{1}{2}(u_t^2 + u_x^2) dx - \int_{\Omega_0} \frac{1}{2}(h^2 + g_x^2) dx,
\end{aligned}$$

where  $\gamma$  is the outer unit normal vector of  $\partial K_\tau$  and has values  $\gamma = (-1, 0)$  on  $\Omega_0$ ,  $\gamma = (1, 0)$  on  $\Omega_\tau$ ,  $\gamma = \frac{1}{\sqrt{2}}(1, -1)$  on  $\Gamma_\tau^1$  and  $\gamma = \frac{1}{\sqrt{2}}(1, 1)$  on  $\Gamma_\tau^2$ .

The right hand side can be estimated by using Young's inequality,

$$\int \int_{K_\tau} u_t f dxdt \leq \frac{1}{2} \int \int_{K_\tau} u_t^2 dxdt + \frac{1}{2} \int \int_{K_\tau} f^2 dxdt.$$

with the above discussions being combined together, we have

$$\int_{\Omega_\tau} (u_t^2 + u_x^2) dx \leq \int_{\Omega_0} (h^2 + g_x^2) dx + \int \int_{K_\tau} u_t^2 dxdt + \int \int_{K_\tau} f^2 dxdt.$$

Let

$$\begin{aligned}
G(\tau) &= \int \int_{K_\tau} (u_t^2 + u_x^2) dxdt = \int_0^\tau \int_{x_0-(t_0-t)}^{x_0+(t_0-t)} (u_t^2 + u_x^2) dxdt, \\
F(\tau) &= \int_{\Omega_0} (h^2 + g_x^2) dx + \int \int_{K_\tau} f^2 dxdt.
\end{aligned}$$

The estimates we have done is equivalent to the following inequality

$$\frac{dG(\tau)}{d\tau} \leq G(\tau) + F(\tau),$$

where  $F(\tau)$  is increasing in  $\tau$ . Then noticing that  $G(0) = 0$ , Gronwall's inequality implies

$$G(\tau) \leq (e^\tau - 1)F(\tau) \leq e^{t_0} F(\tau).$$

□

We can also get the  $L^2$  estimate from the energy estimate.

**Theorem 2.4.** *If  $u \in C^1(\mathbb{R} \times [0, +\infty)) \cap C^2(\mathbb{R} \times (0, +\infty))$  is a solution of (2.13), then  $\forall (x_0, t_0) \in \mathbb{R} \times (0, +\infty)$ ,*

$$\begin{aligned} \int_{\Omega_\tau} u^2(x, \tau) dx &\leq M_1 \left( \int_{\Omega_0} (g^2 + h^2 + g_x^2) dx + \int \int_{K_\tau} f^2 dx dt \right), \\ \int \int_{K_\tau} u^2(x, t) dx dt &\leq M_1 \left( \int_{\Omega_0} (g^2 + h^2 + g_x^2) dx + \int \int_{K_\tau} f^2 dx dt \right), \end{aligned}$$

where  $M_1$  is a constant depends on  $t_0$ ,  $\tau \in [0, t_0]$  and the definitions of domains  $K_\tau$ ,  $\Omega_\tau$  and  $\Omega_0$  are the same as before.

*Proof.* We only need to prove that  $\|u\|_{L^2(\Omega_\tau)}$  and  $\|u\|_{L^2(K_\tau)}$  can be controlled by  $\|u_t\|_{L^2(K_\tau)}$ . In fact,

$$\int_{\Omega_\tau} (u^2(x, \tau) - u^2(x, 0)) dx = \int_{\Omega_\tau} \int_0^\tau \partial_t u^2(x, t) dt dx \leq \int \int_{K_\tau} (u^2 + u_t^2) dx dt.$$

By Gronwall's inequality, we have

$$\begin{aligned} \int_{\Omega_\tau} u^2(x, \tau) dx &\leq C(t_0) \left( \int_{\Omega_0} g^2(x) dx + \int \int_{K_\tau} u_t^2 dx dt \right), \\ \int \int_{K_\tau} u^2(x, t) dx dt &\leq C(t_0) \left( \int_{\Omega_0} g^2(x) dx + \int \int_{K_\tau} u_t^2 dx dt \right). \end{aligned}$$

Thus the  $L^2$  estimate is a direct consequence from energy estimates.  $\square$

**Uniqueness** is a direct corollary of energy estimates. Let  $Q = \mathbb{R} \times (0, +\infty)$ .

**Corollary 2.2.** *If  $u_1$  and  $u_2$  are two  $C^2(Q) \cap C^1(\bar{Q})$  solutions of the Cauchy problem (2.13), then  $u_1 = u_2$  in  $Q$ .*

*Proof.* Let  $w = u_1 - u_2$ , then  $w_{tt} - w_{xx} = 0$  in  $Q$  and  $w|_{t=0} = w_t|_{t=0} = 0$ . Then energy estimates for  $w$  state that  $\forall (x_0, t_0) \in Q$ ,  $0 < \tau < t_0$ , there holds

$$\int_{\Omega_\tau} (w^2 + w_t^2 + w_x^2) dx \leq 0,$$

which implies  $w = 0$  in  $\Omega_\tau$ . Since  $(x_0, t_0)$  is arbitrary, the uniqueness is proved.  $\square$

**Stability** in the sense of  $H^1$ -norm.  $H^1$ -norm of a function is defined by

$$\|u\|_{H^1} = \|u\|_{L^2} + \|\nabla u\|_{L^2}.$$

**Corollary 2.3.** *If  $u_1, u_2$  are  $C^2(Q) \cap C^1(\bar{Q})$  solutions of the Cauchy problem (2.13) with different data  $f_1, g_1, h_1$  and  $f_2, g_2, h_2$  respectively. Then*

$$\|u_1 - u_2\|_{H^1(K_\tau)} \leq M \left( \|g_1 - g_2\|_{H^1(\Omega_0)} + \|h_1 - h_2\|_{L^2(\Omega_0)} + \|f_1 - f_2\|_{L^2(K_\tau)} \right),$$

where

$$\begin{aligned}\|u\|_{H^1(K_\tau)}^2 &= \int \int_{K_\tau} [u^2(x, \tau) + u_t^2(x, \tau) + u_x^2(x, \tau)] dx dt, \\ \|g\|_{H^1(\Omega_0)}^2 &= \int_{\Omega_0} [g^2(x) + g_x^2(x)] dx, \\ \|h\|_{L^2(\Omega_0)}^2 &= \int_{\Omega_0} h^2(x) dx, \\ \|f\|_{L^2(K_\tau)}^2 &= \int \int_{K_\tau} f^2(x, t) dx dt,\end{aligned}$$

and the definitions of domains  $K_\tau$ ,  $\Omega_\tau$  and  $\Omega_0$  are the same as before.

*Proof.* Notice that  $w = u_1 - u_2$  is a solution of

$$\begin{aligned}w_{tt} - w_{xx} &= f_1 - f_2, \quad \text{in } Q, \\ w|_{t=0} &= g_1 - g_2, \quad w_t|_{t=0} = h_1 - h_2,\end{aligned}$$

then the energy estimates and  $L^2$ -estimates directly imply the stability.  $\square$

**2.3. Initial boundary value problem in one dimension.** We consider the initial boundary value problem with homogeneous Dirichlet boundary condition

$$\begin{aligned}u_{tt} - u_{xx} &= 0, \quad x \in (0, 1), t > 0, \\ u|_{t=0} &= g(x), \quad u_t|_{t=0} = h(x), \\ u|_{x=0} &= u|_{x=1} = 0.\end{aligned}\tag{2.14}$$

Other reasonable boundary conditions are Neumann boundary condition and Robin boundary condition. For simplicity, in most of the cases, we only handle the Dirichlet boundary condition.

**2.3.1. Separation of variables (motivation).** The problem (2.14) can be solved by separation of variables. Before stating the rigorous result, we present here a formal calculation as a motivation.

Suppose that our solution has a factorized form  $u(x, t) = X(x)T(t)$ . If we put it into the equation, we get immediately

$$XT'' - X''T = 0 \quad \Rightarrow \quad \frac{X''}{X} = \frac{T''}{T}.$$

Now both sides of this equation are functions of one dimensional variables, but with different variables  $x$  and  $t$ . Once they are equal, they must be the same constant, which is independent of  $x$  and  $t$ . We denote the constant by  $-\lambda$ . By applying the boundary conditions, we have

$$\begin{aligned}X'' + \lambda X &= 0, \quad X(0) = X(1) = 0, \\ T'' + \lambda T &= 0.\end{aligned}$$

Then the solutions would be

$$\begin{aligned} X(x) &= C \cos \sqrt{\lambda}x + D \sin \sqrt{\lambda}x, \\ T(t) &= A \cos \sqrt{\lambda}t + B \sin \sqrt{\lambda}t, \end{aligned}$$

where  $A, B, C$  and  $D$  are constants to be determined later. The reason why  $\lambda$  are nonnegative will be explained later.

Applying the boundary condition for  $X$  yields

$$C = 0, \quad D \sin \sqrt{\lambda} = 0, \quad \Rightarrow \quad \lambda = (n\pi)^2, \quad n = 1, 2, 3, \dots$$

Therefore, for any fixed  $n$ , we have the following solutions with undetermined constants  $A_n$  and  $B_n$ ,

$$\begin{aligned} u_n(x, t) &= (A_n \cos \sqrt{\lambda_n}t + B_n \sin \sqrt{\lambda_n}t) \sin \sqrt{\lambda_n}x \\ &= (A_n \cos n\pi t + B_n \sin n\pi t) \sin n\pi x. \end{aligned}$$

By superposition principle, we know that the finite summation of solutions is still a solution, i.e., for any fixed  $N$ ,

$$u_N(x, t) = \sum_{n=1}^N u_n(x, t) = \sum_{n=1}^N (A_n \cos n\pi t + B_n \sin n\pi t) \sin n\pi x$$

is a solution of the wave equation with Dirichlet boundary condition. If, furthermore, the initial data  $g(x)$  and  $h(x)$  have the same form, say,

$$g(x) = \sum_{n=1}^N g_n \sin n\pi x, \quad h(x) = \sum_{n=1}^N h_n \sin n\pi x,$$

then  $u_N$  must be a solution with this initial data. One could naturally ask how about the case with general initial data? What is the corresponding solution  $u(x, t)$ ? Can we use  $\infty$  to replace  $N$ ? These problems introduce directly to the theory of Fourier series. We first write initial data  $g$  and  $h$  into Fourier series, which can be done for  $L^2$  functions (see the appendix in the end of this chapter), then the corresponding series  $\lim_{N \rightarrow \infty} u_N(x, t)$  can be expected to be a solution for smooth enough initial data. We will prove this rigorously later.

**2.3.2. Generalized Fourier series (Sturm-Liouville theorem).** As a preparation for getting rigorous solution formula, we introduce the so-called Sturm-Liouville problem, which is the theoretical basis of the method of separation of variables. We consider an eigenvalue problem with more general boundary condition (Robin type). Dirichlet and Neumann boundary conditions are merely special cases from Robin boundary condition. Now we consider

$$\begin{aligned} X'' + \lambda X &= 0, \quad 0 < x < 1, \\ -\alpha_1 X'(0) + \beta_1 X(0) &= 0, \quad \alpha_i, \beta_i \geq 0, \\ \alpha_2 X'(1) + \beta_2 X(1) &= 0, \quad \alpha_i + \beta_i > 0. \end{aligned} \tag{2.15}$$

**Theorem 2.5.** (*Sturm-Liouville theorem*)

- (1) All the eigenvalues of (2.15) are nonnegative. In addition, if  $\beta_1 + \beta_2 > 0$ , then all the eigenvalues are positive,
- (2) eigenvalues are countable and increasing to infinity, i.e.,

$$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \quad \lim_{n \rightarrow \infty} \lambda_n = \infty,$$

- (3) eigenfunctions of different eigenvalues are orthogonal in the following sense

$$\int_0^1 X_\lambda X_\mu dx = 0, \quad \text{for } \lambda \neq \mu,$$

- (4)  $\forall f \in L^2(0, 1)$ , it holds that

$$f(x) = \sum_{n=1}^{\infty} C_n X_n(x), \quad \text{in the sense of } L^2(0, 1), \quad C_n = \frac{\int_0^1 f(x) X_n(x) dx}{\int_0^1 X_n^2 dx},$$

or

$$\lim_{n \rightarrow \infty} \|f(x) - f_n(x)\|_{L^2} = 0,$$

where  $\sum_{n=1}^{\infty} C_n X_n(x)$  is called the generalized Fourier series.

*Proof.* We will only proof the first three statements. The last one can be found in functional analysis in the part of compact self-adjoint operators.

- (1) By multiplying the equation by  $X_\lambda$  and integrating it over  $(0, 1)$ , we have

$$X_\lambda X'_\lambda \Big|_0^1 - \int_0^1 (X'_\lambda)^2 dx + \lambda \int_0^1 X_\lambda^2 dx = 0.$$

Boundary conditions show that

$$\begin{aligned} -\alpha_1 X'_\lambda(0) X_\lambda(0) + \beta_1 X_\lambda^2(0) &= 0, & -\alpha_1 (X'_\lambda(0))^2 + \beta_1 X_\lambda(0) X'_\lambda(0) &= 0, \\ \alpha_2 X'_\lambda(1) X_\lambda(1) + \beta_2 X_\lambda^2(1) &= 0, & \alpha_2 (X'_\lambda(1))^2 + \beta_2 X_\lambda(1) X'_\lambda(1) &= 0. \end{aligned}$$

From these, we get

$$\begin{aligned} X'_\lambda(0) X_\lambda(0) &= \frac{1}{\alpha_1 + \beta_1} (\alpha_1 (X'_\lambda(0))^2 + \beta_1 X_\lambda^2(0)), \\ X'_\lambda(1) X_\lambda(1) &= \frac{-1}{\alpha_2 + \beta_2} (\beta_2 X_\lambda^2(1) + \alpha_2 (X'_\lambda(1))^2). \end{aligned}$$

As a consequence, we know the nonnegativity of

$$\lambda \int_0^1 X_\lambda^2 dx = \int_0^1 (X'_\lambda)^2 dx - X'_\lambda(1) X_\lambda(1) + X'_\lambda(0) X_\lambda(0) \geq 0.$$

Thus we have  $\lambda \geq 0$  and furthermore

$$\lambda = 0 \text{ if and only if } X'_\lambda \equiv 0 \text{ and } \frac{\beta_1}{\alpha_1 + \beta_1} X_\lambda^2(0) + \frac{\beta_2}{\alpha_2 + \beta_2} X_\lambda^2(1) = 0,$$

or equivalently

$$X_\lambda \equiv C \text{ and } \left( \frac{\beta_1}{\alpha_1 + \beta_1} + \frac{\beta_2}{\alpha_2 + \beta_2} \right) C^2 = 0.$$

We can see from this expression that if  $\beta_1 + \beta_2 > 0$ , then  $X_\lambda \equiv 0$  which can not be a valid eigenfunction. In the end, we get that in the case of  $\beta_1 + \beta_2 > 0$ ,  $\lambda$  must be positive.

- (2) We have already proved that  $\lambda \geq 0$  and  $\lambda = 0$  iff  $\beta_1 = \beta_2 = 0$ . From  $X'' + \lambda X = 0$ , we know that for all constants  $A$  and  $B$ , the following representation is the solution,

$$X(x) = A \cos \sqrt{\lambda}x + B \sin \sqrt{\lambda}x,$$

$$X'(x) = -A\sqrt{\lambda} \sin \sqrt{\lambda}x + B\sqrt{\lambda} \cos \sqrt{\lambda}x.$$

We will get further information on the constants  $A$  and  $B$  from the boundary conditions. It will be studied in the following three cases:

- (a) **Dirichlet boundary condition**  $\alpha_1 = \alpha_2 = 0$ , then the boundary condition is reduced into  $X(0) = X(1) = 0$ . Simple computations show that  $A = 0$  and  $B \sin \sqrt{\lambda} = 0$ . As a consequence,

$$\lambda_n = (n\pi)^2, \quad X_n(x) = \sin n\pi x, \quad n = 1, 2, \dots$$

Obviously, in this case,  $\lambda_n$  is monotone and increasing to  $\infty$ .

- (b) **Neumann boundary condition**  $\beta_1 = \beta_2 = 0$ , in this case the boundary condition is  $X'(0) = X'(1) = 0$ , which implies that  $B = 0$ ,  $A \sin \sqrt{\lambda} = 0$ . Thus we have

$$\lambda_n = (n\pi)^2, \quad X_n(x) = \cos n\pi x, \quad n = 0, 1, 2, \dots$$

- (c) **Robin boundary condition**  $\alpha_1\beta_2 + \alpha_2\beta_1 > 0$ . From the boundary condition we have

$$0 = \beta_1 A - \alpha_1 B \sqrt{\lambda},$$

$$0 = \beta_2 (A \cos \sqrt{\lambda} + B \sin \sqrt{\lambda}) - \alpha_2 \sqrt{\lambda} (A \sin \sqrt{\lambda} - B \cos \sqrt{\lambda}),$$

which implies that

$$\frac{\alpha_1 \sqrt{\lambda}}{\beta_1} = \frac{A}{B},$$

$$0 = \beta_2 \alpha_1 \sqrt{\lambda} \frac{1}{\tan \sqrt{\lambda}} + \beta_2 \beta_1 - \alpha_1 \alpha_2 \lambda + \beta_1 \alpha_2 \sqrt{\lambda} \frac{1}{\tan \sqrt{\lambda}}.$$

Let  $\xi = \sqrt{\lambda}$ , then we are left to solve the following equation,

$$\tan \xi = \frac{(\beta_2 \alpha_1 + \beta_1 \alpha_2) \xi}{\alpha_1 \alpha_2 \xi^2 - \beta_1 \beta_2}.$$

It can be proved that the sequence of all the possible solutions  $\{\xi_n = \sqrt{\lambda_n}\}$  is monotone increasing and goes to  $\infty$ .

- (3) Let  $\lambda, \mu$  be two different eigenvalues. Multiplying the equation  $X_\lambda'' + \lambda X_\lambda = 0$  by  $X_\mu$  and  $X_\mu'' + \mu X_\mu = 0$  by  $X_\lambda$ , integrating over  $(0, 1)$ , we have

$$\begin{aligned} X_\mu X_\lambda' \Big|_0^1 - \int_0^1 X_\mu' X_\lambda' + \lambda \int_0^1 X_\lambda X_\mu &= 0, \\ X_\lambda X_\mu' \Big|_0^1 - \int_0^1 X_\lambda' X_\mu' + \mu \int_0^1 X_\lambda X_\mu &= 0. \end{aligned}$$

The difference between this two equations shows that

$$(\lambda - \mu) \int_0^1 X_\lambda X_\mu = -X_\mu X_\lambda' \Big|_0^1 + X_\lambda X_\mu' \Big|_0^1.$$

Now the boundary conditions for  $X_\lambda$  and  $X_\mu$  are

$$\begin{aligned} -\alpha_1 X_\lambda'(0) + \beta_1 X_\lambda(0) &= 0, & \alpha_2 X_\lambda'(1) + \beta_2 X_\lambda(1) &= 0 \\ -\alpha_1 X_\mu'(0) + \beta_1 X_\mu(0) &= 0, & \alpha_2 X_\mu'(1) + \beta_2 X_\mu(1) &= 0 \end{aligned}$$

These algebraic systems have non zero solutions, thus the coefficient determinants are 0, i.e.,

$$\begin{vmatrix} X_\lambda'(0) & X_\lambda(0) \\ X_\mu'(0) & X_\mu(0) \end{vmatrix} = 0, \quad \begin{vmatrix} X_\lambda'(1) & X_\lambda(1) \\ X_\mu'(1) & X_\mu(1) \end{vmatrix} = 0.$$

Thus we have

$$(\lambda - \mu) \int_0^1 X_\mu X_\lambda dx = 0.$$

Since  $\lambda \neq \mu$ , we know that  $X_\lambda$  and  $X_\mu$  are orthogonal. □

*Remark 2.3.* When  $\beta_1 = \beta_2 = 0$ , it is Neumann boundary condition. In this case,  $\lambda = 0$  is an eigenvalue, and its eigenfunction is  $X_0 = 1$ .

*Remark 2.4.*  $\{X_n(x)\}$  is a complete orthogonal basis of  $L^2(0, 1)$ . After normalization, it is

$$X_n^*(x) = \frac{X_n(x)}{\|X_n(x)\|_{L^2}}.$$

Then  $\forall f \in L^2(0, 1)$ , the fourier coefficients  $C_n^*$  are

$$C_n^* = \frac{\int_0^1 f(x) X_n(x) dx}{\|X_n(x)\|_{L^2}},$$

which is the inner product of  $f(x)$  and  $X_n^*(x)$ .

2.3.3. *Solution formula by separation of variables.* Now we come back to the solution formula for (2.14) by using the method of separation of variables,

$$u(x, t) = \sum_{n=1}^{\infty} (A_n \cos n\pi t + B_n \sin n\pi t) \sin n\pi x. \quad (2.16)$$

We only need to determine the coefficients  $A_n$  and  $B_n$  by using initial data. Take  $t = 0$  in (2.16), we have

$$u(x, 0) = \sum_{n=1}^{\infty} A_n \sin n\pi x \quad \text{and} \quad u_t(x, 0) = \sum_{n=1}^{\infty} n\pi B_n \sin n\pi x.$$

For the initial data  $g, h \in L^2(0, 1)$ , they have the following Fourier expansion by sine functions

$$\begin{aligned} g(x) &= \sum_{n=1}^{\infty} g_n \sin n\pi x, & g_n &= 2 \int_0^1 g(x) \sin n\pi x dx, \\ h(x) &= \sum_{n=1}^{\infty} h_n \sin n\pi x, & h_n &= 2 \int_0^1 h(x) \sin n\pi x dx, \end{aligned}$$

then the coefficients can be determined by

$$A_n = g_n, \quad B_n = \frac{1}{n\pi} h_n.$$

Thus the solution expression is

$$u(x, t) = \sum_{n=1}^{\infty} (g_n \cos n\pi t + \frac{h_n}{n\pi} \sin n\pi t) \sin n\pi x. \quad (2.17)$$

Now we give a short summary on the three main steps in applying the method of separation of variables:

- (1) let the solution have a factorized form  $u(x, t) = X(x)T(t)$ , and set up the eigenvalue problem,
- (2) solve the eigenvalue problem, and solve the ODE for  $T(t)$ ,
- (3) take a summation of all the factorize-typed solutions, and fix the coefficients by using initial data.

After the above steps, we arrive at a solution formula, there are still a few questions left:

- (1) How can one handle the inhomogeneous equation  $u_{tt} - u_{xx} = f$ ?
- (2) How can one deal with the inhomogeneous boundary conditions?
- (3) Does the formula give us a classical solution? Under what condition is it a classical solution?

We will answer the remaining questions in the following.

**Inhomogeneous equation** We briefly explain how to deal with the inhomogeneous equations and left the details to the readers. Here we use  $(0, l)$  instead of  $(0, 1)$ .

$$\begin{aligned} u_{tt} - u_{xx} &= f(x, t), & x \in (0, l), t > 0, \\ u|_{x=0} &= u|_{x=l} = 0, \\ u|_{t=0} &= g(x), & u_t|_{t=0} = h(x). \end{aligned} \tag{2.18}$$

Firstly we know that the eigenfunctions are  $\sin \frac{n\pi x}{l}$ ,  $n = 1, 2, \dots$ . Then assume that

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi}{l} x, \\ f(x, t) &= \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi}{l} x, \\ g(x) &= \sum_{n=1}^{\infty} g_n \sin \frac{n\pi}{l} x, \\ h(x) &= \sum_{n=1}^{\infty} h_n \sin \frac{n\pi}{l} x. \end{aligned}$$

Then solve the ODE for  $T_n(t)$ ,

$$\begin{aligned} T_n''(t) + \left(\frac{n\pi}{l}\right)^2 T_n(t) &= f_n(t), \\ T_n(0) = g_n, \quad T_n'(0) &= h_n. \end{aligned}$$

One can get that the solution is

$$T_n(t) = g_n \cos \frac{n\pi}{l} t + \frac{l}{n\pi} h_n \sin \frac{n\pi}{l} t + \frac{l}{n\pi} \int_0^t f_n(\tau) \sin \frac{n\pi}{l} (t - \tau) d\tau.$$

Then by replacing  $T_n(t)$  in the solution  $u(x, t)$ , we get the solution for inhomogeneous equation.

### **Inhomogeneous boundary conditions**

The problem with inhomogeneous boundary condition is

$$\begin{aligned} u_{tt} - u_{xx} &= f, & x \in (0, l), t > 0, \\ u|_{x=0} &= u_0(t), & u|_{x=l} = u_1(t), \\ u|_{t=0} &= g(x), & u_t|_{t=0} = h(x). \end{aligned} \tag{2.19}$$

We will use homogenization technic. In other words, by introducing a new function  $v(x, t)$  such that the homogeneous boundary conditions work for  $v(x, t)$ . More precisely, let

$$v(x, t) := u(x, t) - \frac{x}{l} u_1(t) - \frac{l-x}{l} u_0(t),$$

then  $v(x, t)$  solves the following initial boundary value problem

$$\begin{aligned} v_{tt} - v_{xx} &= f(x, t) - \frac{x}{l}u_1'' - \frac{l-x}{l}u_0'', \\ v|_{x=0} &= v|_{x=l} = 0, \\ v|_{t=0} &= g(x) - \frac{x}{l}u_1(0) - \frac{l-x}{l}u_0(0), \\ v_t|_{t=0} &= h(x) - \frac{x}{l}u_1'(0) - \frac{l-x}{l}u_0'(0). \end{aligned}$$

By the method of dealing with inhomogeneous equations, we can get a formula for  $v(x, t)$ , which gives directly the expression of  $u(x, t)$ .

2.3.4. *Existence of solution for (2.14).* Under what conditions for the given data is the series in (2.17) exactly the classical solution of (2.14)?

If we can prove that  $u$  is at least twice differentiable in both  $x$  and  $t$ , then all the previous computations can go through, which means that it is indeed a classical solution. Therefore, we are left to prove that

$$\sum_{n=1}^{\infty} u_n, \quad \sum_{n=1}^{\infty} Du_n, \quad \sum_{n=1}^{\infty} D^2u_n$$

are uniformly convergent in  $(0, 1) \times (0, T)$ .

In order to obtain the classical solution of (2.14), we require the following compatibility conditions

$$g(0) = g(1) = 0, \quad h(0) = h(1) = 0, \quad g''(0) = g''(1) = 0. \quad (2.20)$$

**Theorem 2.6.** *If  $g \in C^3[0, 1]$ ,  $h \in C^2[0, 1]$  and they satisfy the compatibility condition (2.20), then  $u(x, t)$  given by (2.17)  $= \sum_{n=1}^{\infty} u_n(x, t) \in C^2(\bar{Q})$  is a solution of (2.14).*

*Proof.* Doing integration by parts on the coefficients of  $g$  and  $h$  by using compatibility conditions, we have

$$\begin{aligned} \frac{h_n}{n\pi} &= \frac{2}{n\pi} \int_0^1 h(x) \sin n\pi x dx = -\frac{2}{(n\pi)^3} \int_0^1 h''(x) \sin n\pi x dx =: -\frac{2}{(n\pi)^3} a_n, \\ g_n &= 2 \int_0^1 g(x) \sin n\pi x dx = \frac{2}{(n\pi)^3} \int_0^1 g'''(x) \cos n\pi x dx =: \frac{2}{(n\pi)^3} b_n. \end{aligned}$$

Therefore

$$u(x, t) = \sum_{n=1}^{\infty} \left( \frac{2}{(n\pi)^3} b_n \cos n\pi t - \frac{2}{(n\pi)^3} a_n \sin n\pi t \right) \sin n\pi x.$$

Moreover the following estimates hold

$$\begin{aligned} |u_n| &\leq \frac{C}{n^3}, \quad |Du_n| \leq \frac{C}{n^2}, \\ |D^2u_n| &\leq \frac{C}{n} (|a_n| + |b_n|) \leq C \left( \frac{1}{n^2} + |a_n|^2 + |b_n|^2 \right), \end{aligned}$$

where the right hand side of the last inequality can be bounded by Bessel's inequality,

$$\sum_{n=1}^{\infty} |a_n|^2 \leq 2 \int_0^1 |h''|^2 dx, \quad \sum_{n=1}^{\infty} |b_n|^2 \leq 2 \int_0^1 |g'''|^2 dx.$$

□

2.3.5. *Uniqueness and Stability — Energy estimates.* Let  $Q_\tau = (0, 1) \times (0, \tau)$ . We have the following energy estimates for initial boundary value problem of wave equation (2.14) in  $Q_\tau$ . Then uniqueness and stability can be obtained from that.

**Theorem 2.7.** *Assume  $u \in C^2(Q_\tau) \cap C^1(\bar{Q}_\tau)$  is a solution of (2.14), then*

$$\int_0^1 (u^2 + u_t^2 + u_x^2) dx \leq M \left( \int_0^1 (h^2 + g^2 + g_x^2) dx + \int_{Q_\tau} f^2 dx dt \right).$$

*Proof.* Multiplying the wave equation by  $u_t$  and integrating it over  $Q_\tau$  gives

$$\int_{Q_\tau} \frac{\partial}{\partial t} (u_t^2 + u_x^2) \leq \int_{Q_\tau} f^2 + \int_{Q_\tau} u_t^2.$$

Notice that  $\int_{Q_\tau} = \int_0^\tau \int_0^1$ , we have

$$\int_0^1 (u_t^2 + u_x^2)|_{t=\tau} \leq \int_0^1 (h^2 + g_x^2) + \int_{Q_\tau} f^2 + \int_{Q_\tau} u_t^2.$$

By Gronwall's inequality,

$$\int_0^1 (u_t^2 + u_x^2)|_{t=\tau} \leq M \left( \int_0^1 (h^2 + g_x^2) + \int_{Q_\tau} f^2 \right).$$

Similar to the discussion in Cauchy problem, we have the  $L^2$  estimates.

□

2.3.6. *Resonance.* In the solution formula,  $u(x, t) = \sum_{n=1}^{\infty} u_n(x, t)$ , the  $n$ -th wave is a wave that remains in a constant position in the following sense

$$\begin{aligned} u_n(x, t) &= (g_n \cos n\pi t + \frac{h_n}{n\pi} \sin n\pi t) \sin n\pi x \\ &= (g_n^2 + (h_n/n\pi)^2)^{\frac{1}{2}} \sin n\pi x \left( \frac{g_n}{(g_n^2 + (h_n/n\pi)^2)^{\frac{1}{2}}} \cos n\pi t \right. \\ &\quad \left. + \frac{h_n/n\pi}{(g_n^2 + (h_n/n\pi)^2)^{\frac{1}{2}}} \sin n\pi t \right) \\ &= (g_n^2 + (h_n/n\pi)^2)^{\frac{1}{2}} \sin n\pi x \cdot \sin(n\pi t + \alpha_n) \\ &= N_n \cdot \sin n\pi x \cdot \sin(n\pi t + \alpha_n), \end{aligned}$$

where  $\tan \alpha_n = \frac{n\pi g_n}{h_n}$ ,  $N_n = (g_n^2 + (h_n/n\pi)^2)^{\frac{1}{2}}$ . Usually  $N_1 \gg N_n, \forall n \neq 1$ .

Consider the following initial boundary value problem

$$\begin{aligned} u_{tt} - u_{xx} &= A(x) \sin \omega t, \quad x \in (0, 1), t > 0, \\ u|_{x=0,1} &= 0, \\ u|_{t=0} &= u_t|_{t=0} = 0. \end{aligned}$$

Compatibility condition  $A(0) = A(1) = 0$  is needed for the existence of classical solution. We assume  $A \in C^1$ . The solution formula from separation of variables is

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} \frac{1}{n\pi} \int_0^t f_n(\tau) \sin n\pi(t - \tau) d\tau \cdot \sin n\pi x \\ &= \sum_{n=1}^{\infty} \frac{a_n}{n\pi} \sin n\pi x \int_0^t \sin \omega \tau \cdot \sin n\pi(t - \tau) d\tau, \end{aligned}$$

where

$$a_n = 2 \int_0^1 A(x) \sin n\pi x dx.$$

If we calculate further, we will see

$$\begin{aligned} & \int_0^t \sin \omega \tau \cdot \sin n\pi(t - \tau) d\tau \\ &= \int_0^t -\frac{1}{2} (\cos(n\pi t + (\omega - n\pi)\tau) - \cos(-(\omega + n\pi)\tau + n\pi t)) d\tau \\ &= \frac{1}{2} \int_0^t \cos((\omega + n\pi)\tau - n\pi t) d\tau - \frac{1}{2} \int_0^t \cos((\omega - n\pi)\tau + n\pi t) d\tau \\ &\stackrel{\omega \neq n\pi}{=} \frac{1}{2(\omega + n\pi)} \sin((\omega + n\pi)\tau - n\pi t) \Big|_0^t - \frac{1}{2(\omega - n\pi)} \sin((\omega - n\pi)\tau + n\pi t) \Big|_0^t \\ &\stackrel{\omega \neq n\pi}{=} \frac{1}{2(\omega + n\pi)} (\sin \omega t + \sin n\pi t) - \frac{1}{2(\omega - n\pi)} (\sin \omega t - \sin n\pi t). \end{aligned}$$

If  $\omega = k\pi$  for some  $k$ , then

$$\begin{aligned} u_k(x, t) &= \frac{a_k}{k\pi} \left( \frac{\sin k\pi t}{k\pi + k\pi} - \frac{1}{2} \int_0^t \cos k\pi t d\tau \right) \sin k\pi x \\ &= \left( \frac{a_k}{2(k\pi)^2} \sin k\pi t - \frac{a_k}{2k\pi} t \cdot \cos k\pi t \right) \sin k\pi x. \end{aligned}$$

Thus in the case of  $\omega = k\pi$ , we have

$$\begin{aligned} u(x, t) &= \sum_{n \neq k} \frac{a_n}{n\pi} \left( \frac{1}{2(\omega + n\pi)} (\sin \omega t + \sin n\pi t) - \frac{1}{2(\omega - n\pi)} (\sin \omega t - \sin n\pi t) \right) \sin n\pi x \\ &\quad + \left( \frac{a_k}{2(k\pi)^2} \sin k\pi t - \frac{a_k}{2k\pi} t \cdot \cos k\pi t \right) \sin k\pi x \end{aligned}$$

Therefore there exists a sequence of time  $t_m \rightarrow \infty$  such that the  $u_k(x, t)$  is unbounded for big  $m$ .

**2.4. Appendix-On Fourier Series.**  $\forall f \in L^1(-l, l)$ , it can be written into a series by using trigonometric functions

$$f(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l} \right),$$

where

$$A_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx, \quad n = 0, 1, 2, \dots,$$

$$B_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx, \quad n = 1, 2, \dots$$

are called Fourier coefficients of  $f$ .

If  $f(x)$  is an even function, then  $B_n = 0$ , and

$$f(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}, \quad A_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx.$$

If  $f(x)$  is an odd function, then  $A_n = 0$ , and

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}, \quad B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx.$$

For any fixed  $N \geq 1$ ,  $(S_N f)(x) = \frac{A_0}{2} + \sum_{n=1}^N \left( A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l} \right)$  is called trigonometric polynomial.

$\sin \frac{n\pi x}{l}$ ,  $\cos \frac{m\pi x}{l}$ ,  $n, m = 1, 2, \dots$  are orthogonal in the following sense

$$\frac{1}{l} \int_{-l}^l \cos \frac{m\pi x}{l} \cos \frac{n\pi x}{l} dx = \delta_{mn},$$

$$\frac{1}{l} \int_{-l}^l \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l} dx = \delta_{mn},$$

$$\frac{1}{l} \int_{-l}^l \sin \frac{m\pi x}{l} \cos \frac{n\pi x}{l} dx = 0.$$

Moreover,  $\{1, \sqrt{2} \cos \frac{n\pi x}{l}, \sqrt{2} \sin \frac{n\pi x}{l}\}_{n=1}^{\infty}$  is an orthonormal basis in  $L^2(-l, l)$ , where the inner product in  $L^2(-l, l)$  is defined by  $\frac{1}{2l} \int_{-l}^l f(x) \bar{g}(x) dx$ ,  $\forall f, g \in L^2$ .

Here we list several useful facts for trigonometric series that we are going to use in this course.

**Theorem 2.8.** (Convergence in  $L^2$  norm)

$$\lim_{N \rightarrow \infty} \|f(x) - (S_N f)(x)\|_{L^2} = 0, \quad \text{for } f \in L^2(-l, l).$$

**Theorem 2.9.** (*Bessel's inequality*) For  $f \in L^2(-l, l)$ , it holds

$$\frac{A_0^2}{2} + \sum_{n=1}^{\infty} (A_n^2 + B_n^2) \leq \frac{1}{l} \int_{-l}^l f^2 dx.$$

**Theorem 2.10.** (*Parseval's equality*) For  $f \in L^2(-l, l)$ , it holds

$$\frac{A_0^2}{2} + \sum_{n=1}^{\infty} (A_n^2 + B_n^2) = \frac{1}{l} \int_{-l}^l f^2 dx.$$

2.5. \*\*\***Generalized solution.** It happens quite often that in some of the physical models one cannot find the solution in the classical sense. For example, in the model for the vibrating string, the initial data can be piecewise differentiable. A typical example is  $g(x) = x$  for  $x \in [0, 1/2]$  and  $g(x) = 1 - x$  for  $x \in [1/2, 1]$ . Now  $g$  is no longer a differentiable function, but the problem should still have a “reasonable” solution. One must search for the solution in a broader class.

In this subsection, we introduce the basic knowledge of weak solution for the following initial boundary value problem

$$\begin{aligned} u_{tt} - u_{xx} &= f, & Q_T &= (0, 1) \times (0, T) \\ u|_{x=0} &= u|_{x=1} = 0, \\ u|_{t=0} &= g, & u_t|_{t=0} &= h. \end{aligned}$$

Here we give a motivation in defining the solution in a more generalized sense.

The motivation is the following. We multiply the equation by  $\varphi$  and integrate on the domain  $Q_T$ ,

$$\int_{Q_T} (u_{tt} - u_{xx})\varphi = \int_{Q_T} f\varphi.$$

Integration by parts shows that

$$\int_{Q_T} u(\varphi_{tt} - \varphi_{xx}) + \int_0^1 u_t\varphi|_0^T dx - \int_0^1 u\varphi_t|_0^T dx + \int_0^T u_x\varphi|_0^1 dt - \int_0^T u\varphi_x|_0^1 dt = \int_{Q_T} f\varphi.$$

If we choose the test function  $\varphi$  to be the one that satisfies the boundary conditions

$$\varphi|_{t=T} = \varphi_t|_{t=T} = 0, \quad \varphi|_{x=0,1} = 0,$$

we will have

$$\int_{Q_T} u(\varphi_{tt} - \varphi_{xx}) + \int_0^1 h\varphi(x, 0) - \int_0^1 g\varphi_t(x, 0) = \int_{Q_T} f\varphi.$$

Notice that in the above equation, we don't require any smoothness of  $u$ , which means we can define the solution by using the above equation, which is of the integral form.

Now we will fix our test function in the set

$$\mathcal{D} = \{\varphi \in C^2(\bar{Q}_T) | \varphi(x, T) = \varphi_t(x, T) = 0, \varphi|_{x=0,1} = 0\}.$$

**Definition 1.** If  $u \in C(\bar{Q})$  and  $\forall \varphi \in \mathcal{D}$ , there holds

$$\int_{Q_T} u(\varphi_{tt} - \varphi_{xx}) - \int_0^1 h\varphi(x, 0) + \int_0^1 g\varphi_t(x, 0) = \int_{Q_T} f\varphi, \quad (2.21)$$

then we call  $u$  is a weak solution of (2.18).

*Remark 2.5.* Due to the motivations, we know that classical solutions must be weak solutions.

**Theorem 2.11.** (*Uniqueness of weak solution*) If  $u_1$  and  $u_2$  are two weak solutions of (2.18), then  $u_1 = u_2$  in  $Q_T$ .

*Proof.* By definition, we know that  $\forall \varphi \in \mathcal{D}$ ,

$$\int_{Q_T} (u_1 - u_2)(\varphi_{tt} - \varphi_{xx}) = 0.$$

"We want to show that  $\forall \psi \in C_0^\infty(Q_T)$

$$\begin{aligned} \varphi_{tt} - \varphi_{xx} &= \psi, \\ \varphi|_{x=0,1} &= 0, \varphi|_{t=T} = 0, \varphi_t|_{t=T} = 0 \end{aligned}$$

has a solution. This is a backward wave equation, by changing variable  $\tau = T - t$ , let  $\bar{\varphi}(x, \tau) = \varphi(x, t)$ , the problem will change into

$$\begin{aligned} \bar{\varphi}_{tt} - \bar{\varphi}_{xx} &= \psi(x, T - \tau), \\ \bar{\varphi}|_{x=0,1} &= 0, \bar{\varphi}|_{t=T} = 0, \bar{\varphi}_t|_{t=T} = 0. \end{aligned}$$

Since  $\psi \in C_0^\infty(Q_T)$  satisfies the compatibility conditions, from existence theory, we know that this problem has a solution  $\varphi(x, t) \in C^2(\bar{Q}_T)$ . Since  $\psi$  is arbitrary, we know that  $u_1 = u_2$ .  $\square$

**Theorem 2.12.** (*\*\*\*Stability of weak solution*) If  $f \equiv 0$ , then

$$\int_{Q_T} (u_1 - u_2)^2 \leq C \left( \int_0^1 |g_1 - g_2|^2 + \int_0^1 |h_1 - h_2|^2 \right).$$

*Proof.*  $\forall \varphi \in C_0^\infty(Q_T)$ , by the definition of weak solution, we have

$$\int_{Q_T} (u_1 - u_2)(\varphi_{tt} - \varphi_{xx}) - \int_0^1 (h_1 - h_2)\varphi(x, 0) + \int_0^1 (g_1 - g_2)\varphi_t(x, 0) = 0. \quad (2.22)$$

Now we consider problem

$$\begin{aligned} \varphi_{tt} - \varphi_{xx} &= u_1^\varepsilon - u_2^\varepsilon, \\ \varphi|_{x=0,1} &= 0, \varphi|_{t=T} = 0, \varphi_t|_{t=T} = 0, \end{aligned}$$

where  $u_1^\varepsilon, u_2^\varepsilon \in C_0^\infty(Q_T)$  and  $u_i^\varepsilon \rightarrow u_i$  in  $L^2(Q_T)$ ,  $i = 1, 2$ . By changing time variable  $\tau = T - t$ , we have

$$\begin{aligned} \bar{\varphi}_{tt} - \bar{\varphi}_{xx} &= (u_1^\varepsilon - u_2^\varepsilon)(x, T - \tau), \\ \bar{\varphi}|_{x=0,1} &= 0, \bar{\varphi}|_{t=T} = 0, \bar{\varphi}_t|_{t=T} = 0. \end{aligned}$$

Then by the existence of classical solution, we know that the above problem has a solution  $\bar{\varphi}(x, \tau) \in C^2(\bar{Q}_T)$  and the energy estimates hold, i.e.,

$$\int_0^1 (|\bar{\varphi}|^2 + |\bar{\varphi}_\tau|^2) \Big|_{\tau=T} \leq C \|u_1^\varepsilon - u_2^\varepsilon\|_{L^2(Q_T)}^2 \leq C \|u_1 - u_2\|_{L^2(Q_T)}^2.$$

Therefore putting them together into (2.22), we have

$$\int_{Q_T} (u_1 - u_2)(u_1^\varepsilon - u_2^\varepsilon) \leq \|h_1 - h_2\|_{L^2} \cdot \|\varphi\|_{L^2} + \|g_1 - g_2\|_{L^2} \cdot \|\varphi_t\|_{L^2}.$$

Taking  $\varepsilon \rightarrow 0$ , we have

$$\int_{Q_T} (u_1 - u_2)^2 \leq C(\|h_1 - h_2\|_{L^2} + \|g_1 - g_2\|_{L^2}) \cdot \|u_1 - u_2\|_{L^2}.$$

This completes the proof.  $\square$

**Theorem 2.13.** (*Existence of weak solution*) Let  $g \in C[0, 1]$ ,  $g(0) = g(1) = 0$ ,  $g'$  and  $h$  are piecewise continuous in  $[0, 1]$ , then

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t)$$

is a weak solution of (2.14).

*Proof.* Fix an integer  $N > 0$ , let

$$G_N(x) = \sum_{n=1}^N g_n \sin n\pi x, \quad H_N(x) = \sum_{n=1}^N h_n \sin n\pi x.$$

Now we consider the approximation problem

$$\begin{aligned} \partial_{tt} u_N - \partial_{xx} u_N &= 0, & \text{in } (0, 1) \times (0, T], \\ u_N|_{x=0,1} &= 0, \\ u_N|_{t=0} &= G_N, \quad \partial_t u_N|_{t=0} = H_N. \end{aligned} \tag{2.23}$$

Then by separation of variables,

$$u_N = \sum_{n=1}^N \left( g_n \cos n\pi t + \frac{h_n}{n\pi} \sin n\pi t \right) \sin n\pi x$$

is a classical solution of (2.23).

For any test function  $\varphi \in \mathcal{D}$ , we have

$$\int_{Q_T} u_N (\varphi_{tt} - \varphi_{xx}) + \int_0^1 G_N(x) \varphi_t(x, 0) - \int_0^1 H_N(x) \varphi(x, 0) = 0. \tag{2.24}$$

If  $g(x), h(x) \in L^2(0, 1)$ , we know that

$$G_N(x) \rightarrow g(x), \quad H_N(x) \rightarrow h(x), \quad \text{in } L^2.$$

Moreover,

$$\begin{aligned} |u_N| &\leq \sum_{n=1}^N \left( |g_n| + \left| \frac{h_n}{n\pi} \right| \right) \leq \sum_{n=1}^N \frac{C}{n} (|(g')_n| + |h_n|) \\ &\leq C \sum_{n=1}^N \left( \frac{1}{n^2} + |(g')_n|^2 + |h_n|^2 \right). \end{aligned}$$

So we have that

$$u_N \rightarrow u = \sum_{n=1}^{\infty} u_n, \quad \text{uniformly in } (0, 1).$$

Now we take limit  $N \rightarrow \infty$  in (2.25), it follows

$$\int_{Q_T} u(\varphi_{tt} - \varphi_{xx}) + \int_0^1 g(x)\varphi_t(x, 0) - \int_0^1 h(x)\varphi(x, 0) = 0.$$

□

*Remark 2.6.* In the above proof, the condition that  $g'$  and  $h \in L^2(0, 1)$  is sufficient to guarantee the existence of weak solution.

## 2.6. Problems.

- (1) Verify that  $u(x, t) = \frac{F(x - at) + G(x + at)}{h - x}$  is a solution of

$$\left(1 - \frac{x}{h}\right)^2 \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial}{\partial x} \left[ \left(1 - \frac{x}{h}\right)^2 \frac{\partial u}{\partial x} \right]$$

where  $h > 0$ ,  $a > 0$  are constants,  $F, G$  are any function in  $C^2$ .

- (2) (a) Show the general solution of the PDE  $u_{xy} = 0$  is  $u(x, t) = F(x) + G(y)$  for arbitrary function  $F, G$ .  
 (b) Using the change of variables  $\xi = x + t$ ,  $\eta = x - t$ , show  $u_{tt} - u_{xx} = 0$  if and only if  $u_{\xi\eta} = 0$ .  
 (c) Use the above two facts to derive d'Alembert's formula.  
 (3) Give energy estimates for half-line problem and the Cauchy problem in Multi-D case.  
 (4) (Equal partition of energy) Suppose that  $u \in C^2(\mathbb{R} \times [0, \infty))$  is a solution of the following Cauchy problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0, & (x, t) &\in \mathbb{R} \times (0, \infty), \\ u|_{t=0} &= g, & u_t|_{t=0} &= h, & x &\in \mathbb{R}, \end{aligned}$$

where  $g$  and  $h$  have compact support. Let kinetic energy be  $k(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x, t) dx$ ,

potential energy be  $p(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x, t) dx$ . Try to prove

- (a)  $k(t) + p(t)$  is a constant independent of  $t$ .  
 (b)  $k(t) = p(t)$  for large enough  $t$ .

(5)

$$\begin{aligned} u_{tt} - u_{xx} &= 0, & x \in (0, +\infty), t \in (0, +\infty), \\ u|_{x=0} &= \cos \omega t, \\ u|_{t=0} &= Ae^{-x^2}, & u_t|_{t=0} = 0. \end{aligned}$$

Find the condition for  $A$  and  $\omega$  such that solution  $u \in C^2(\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+)$ , and give this solution formula.

(6) If  $u$  is a classical solution of

$$\begin{aligned} u_{tt} - u_{xx} &= 0, & x \in (0, 1), t \in (0, +\infty), \\ u|_{x=0} &= u|_{x=1} = 0, \\ u|_{t=0} &= 0, & u_t|_{t=0} = x^2(1-x), \end{aligned}$$

try to compute the limit

$$\lim_{t \rightarrow +\infty} \int_0^1 (u_t^2 + u_x^2) dx.$$

(7) Solve the eigenvalue problem

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, & x \in (0, l), \\ X(0) &= X'(l) = 0. \end{aligned}$$

(8)

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, & x \in (0, 1), \\ X'(0) + X(0) &= 0, & X(1) = 0. \end{aligned}$$

- (a) Find an eigenfunction with eigenvalue zero. Call it  $X_0(x)$ .
- (b) Find an equation for the positive eigenvalues  $\lambda = \beta^2$ .
- (c) Show graphically from part (8b) that there are an infinite number of positive eigenvalues.
- (d) Is there a negative eigenvalue?

(9) Apply separation of variables to get formal solution of

$$\begin{aligned} u_{tt} - u_{xx} &= 0, & (x, t) \in (0, 1) \times (0, \infty), \\ u_x|_{x=0} &= A \sin \omega t, u|_{x=1} = 0, & t \geq 0, \\ u|_{t=0} &= 1, & u_t|_{t=0} = 0, & x \in [0, 1]. \end{aligned}$$

(10)

$$\begin{aligned} u_{tt} - u_{xx} &= 0, & x \in (0, 1), t \in (0, +\infty), \\ u|_{x=0} &= u|_{x=1} = 0, \\ u|_{t=0} &= \alpha x^4 + \beta x^3 + \sin x, & u_t|_{t=0} = \gamma \cos x. \end{aligned}$$

Solve the problem and give the conditions on  $\alpha$ ,  $\beta$  and  $\gamma$  such that the solution you gave is a classical one.

- (11) Find the solution of initial boundary values for heat equation by separation of variables.

$$\begin{aligned}u_t - u_{xx} &= \sin x\pi, & x \in (0, 1), t \in (0, +\infty), \\u|_{x=0} &= u|_{x=1} = 0, \\u|_{t=0} &= 0.\end{aligned}$$

- (12) **Discussions** One can get solution of (2.14) by d'Alembert' formula and Fourier series. Are they the same?

### 3. HEAT EQUATION

We will first give an introduction for Fourier transform and distribution, together with a list of their properties that will be used within this course. The solution for Cauchy problem of heat equation will be given by using heat kernel. Furthermore, the solution formula for half space problem will be derived by using Green's function. As further understanding of the Fourier series and energy methods, the initial boundary value problem for heat equation will be also be briefly studied. We will also introduce a new method, the maximum principle, for parabolic and elliptic type of equations. It provides the first stage a priori estimate and also can be used to study the uniqueness and stability of the solutions. In the end, we will study the long time behavior of the solution of heat equation, with Dirichlet and Neumann boundary conditions separately.

#### 3.1. A short introduction on Fourier transform and distribution.

3.1.1. *Fourier transform.* Let's first recall the Fourier series.  $\forall f \in L^1(-l, l)$ , its Fourier series is defined by

$$f(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l} \right),$$

where

$$A_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx, \quad n = 0, 1, 2, \dots,$$

$$B_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx, \quad n = 1, 2, \dots$$

By Euler's formula, we can change the items in the summation into

$$A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l} = A'_n e^{i \frac{n\pi x}{l}} + B'_n e^{-i \frac{n\pi x}{l}}.$$

Thus the Fourier series can be rewritten into

$$f(x) \sim \sum_{-\infty}^{\infty} a_n e^{i \frac{n\pi x}{l}}, \quad a_n = \frac{1}{2l} \int_{-l}^l f(x) e^{-i \frac{n\pi x}{l}} dx,$$

i.e.,

$$f(x) \sim \frac{1}{2l} \sum_{-\infty}^{\infty} \int_{-l}^l f(y) e^{-i \frac{n\pi y}{l}} dy e^{i \frac{n\pi x}{l}}.$$

Now let  $k_n = \frac{n\pi}{l}$ , the formula is reduced into

$$f(x) \sim \frac{1}{2\pi} \sum_{-\infty}^{\infty} \int_{-l}^l f(y) e^{-ik_n y} dy e^{ik_n x \frac{\pi}{l}}.$$

Formally letting  $l \rightarrow \infty$ , one could expect that

$$f(x) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) e^{-iky} dy e^{ikx} dk.$$

These formal computations give the motivation in the definition of Fourier transform on  $\mathbb{R}$ .

**Definition 2.**  $\forall f \in L^1(\mathbb{R})$ , its Fourier transform is defined by

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx.$$

A result which can be obtained directly from the definition is that  $\hat{f}(k) \in L^\infty(\mathbb{R})$ , i.e.,

$$|\hat{f}(k)| = \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \right| \leq \frac{1}{\sqrt{2\pi}} \|f\|_{L^1}.$$

Therefore the definition tells that Fourier transform is a continuous linear mapping from  $L^1$  to  $L^\infty$ . Moreover, the following result holds

**Theorem 3.1.** *If  $f \in L^1(\mathbb{R})$ , then  $\hat{f}(k)$  is uniformly continuous in  $\mathbb{R}$ .*

*Proof.* We prove it directly by using the definition of uniformly continuous.  $\forall \varepsilon > 0$ ,  $f \in L^1(\mathbb{R})$  implies that  $\exists A > 0$ , such that

$$\frac{1}{\sqrt{2\pi}} \int_{|x|>A} 2|f| dx \leq \frac{\varepsilon}{2}.$$

$\forall 0 < h < \frac{\sqrt{2\pi}\varepsilon}{4A\|f\|_{L^1}}$ , we have

$$\begin{aligned} |\hat{f}(k+h) - \hat{f}(k)| &= \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} f(x) e^{-ixk} [e^{-ixh} - 1] dx \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{|x|>A} 2|f| dx + \frac{1}{\sqrt{2\pi}} \int_{-A}^A |x| \cdot |h| \cdot |f| dx \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

*Remark 3.1.* Similarly, one can define Fourier transform in multi-D case,  $\forall f \in L^2(\mathbb{R}^n)$ ,

$$\hat{f}(k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) e^{-ik \cdot x} dx,$$

where  $k \cdot x = \sum_{i=1}^n k_i x_i$ . It is also a continuous linear mapping from  $L^1(\mathbb{R}^n)$  to  $L^\infty(\mathbb{R}^n)$ .

We list here a series of properties of Fourier transform without proof.

- (1)  $(\partial_{x_j} f)^\wedge(k) = ik_j \hat{f}(k)$ , and  $(\partial^\alpha f)^\wedge(k) = i^{|\alpha|} k^\alpha \hat{f}(k)$ ,
- (2)  $(x_j f)^\wedge(k) = i \partial_{k_j} \hat{f}(k)$ , and  $(x^\alpha f)^\wedge(k) = i^{|\alpha|} \partial^\alpha \hat{f}(k)$ ,
- (3)  $f(x-a)^\wedge(k) = e^{-ia \cdot k} \hat{f}(k)$ ,
- (4)  $(f(\lambda x))^\wedge(k) = \frac{1}{|\lambda|^n} \hat{f}\left(\frac{k}{\lambda}\right)$ ,  $\forall \lambda \neq 0$ ,

$$(5) (f * g)^\wedge(k) = (2\pi)^{\frac{n}{2}} \hat{f}(k) \hat{g}(k).$$

**Example 1.** Fourier transform of Gaussian  $e^{-x^2}$  in 1-d is  $\frac{1}{\sqrt{2}} e^{-\frac{k^2}{4}}$ . More general case in multi-dimension is

$$(e^{-A|x|^2})^\wedge(k) = \prod_1^n (e^{-Ax_i^2})^\wedge(k_i) = \frac{1}{(2A)^{\frac{n}{2}}} e^{-\frac{|k|^2}{4A}}, \quad \forall A > 0.$$

The **inverse Fourier transform** can be formally given by

$$\check{f}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(k) e^{ik \cdot x} dk.$$

The following theorem assures that the inverse Fourier transform is also well-defined.

**Theorem 3.2.** If  $f \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$ , then

$$\lim_{N \rightarrow \infty} \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-N}^N \hat{f}(k) e^{ikx} dk = f(x).$$

*Proof.* \*(For those who are interested) We know that  $\hat{f}(k)$  is uniformly bounded and continuous in  $k \in \mathbb{R}$ , by the definition of Fourier transform, we have

$$\begin{aligned} & \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-N}^N \hat{f}(k) e^{ikx} dk \\ &= \frac{1}{2\pi} \int_{-N}^N \int_{-\infty}^{\infty} f(y) e^{-iky} dy e^{ikx} dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_{-N}^N e^{ik(x-y)} dk \right) f(y) dy, \end{aligned}$$

where

$$\int_{-N}^N e^{ik(x-y)} dk = 2 \frac{\sin N(x-y)}{x-y}.$$

This is similar to the Dirichlet kernel, therefore one can expect that the whole integral will converge to  $f(x)$  as  $N \rightarrow \infty$ . Next we will prove it in detail.

Changing variable  $z = y - x$  gives

$$\begin{aligned} & \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-N}^N \hat{f}(k) e^{ikx} dk \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin N(x-y)}{x-y} f(y) dy = \frac{1}{\pi} \int_{-\infty}^{\infty} f(z+x) \frac{\sin Nz}{z} dz. \end{aligned}$$

Now we separate the integral on  $\mathbb{R}$  into two parts  $I_1 = \int_{|z| \leq M}$  and  $I_2 = \int_{|z| \geq M}$ , where  $M$  is to be determined later. In the following, we will estimate  $I_1$  by using Riemann' lemma, while the estimate for  $I_2$  by  $1/M$ .

$\forall \varepsilon > 0$ , choose  $M = \frac{2\|f\|_{L^1}}{\pi\varepsilon}$ , we have

$$I_2 = \frac{1}{\pi} \int_{|z| \geq M} f(z+x) \frac{\sin Nz}{z} dz \leq \frac{1}{\pi M} \|f\|_{L^1} = \frac{\varepsilon}{2}.$$

The way to estimate  $I_1$  is by using

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi,$$

from which we know that  $\exists N \geq 0$  s.t.

$$\left| \frac{f(x)}{\pi} \int_{-MN}^{MN} \frac{\sin z}{z} dz - f(x) \right| \leq \frac{\varepsilon}{4}.$$

Now  $I_1$  can be estimated in the following way,

$$\begin{aligned} I_1 &= \frac{1}{\pi} \int_{|z| \leq M} f(z+x) \frac{\sin Nz}{z} dz \\ &= \frac{1}{\pi} \int_{|z| \leq M} \frac{f(z+x) - f(x)}{z} \sin Nz dz + \frac{f(x)}{\pi} \int_{|z| \leq M} \frac{\sin Nz}{z} dz \\ &= \frac{1}{\pi} \int_{|z| \leq M} \int_0^1 f'(x + \tau z) d\tau \sin Nz dz + \frac{f(x)}{\pi} \int_{|z| \leq M} \frac{\sin Nz}{z} dz \\ &\leq \frac{\|f'\|_{L^\infty}}{\pi} \int_{|z| \leq M} \sin Nz dz + \frac{f(x)}{\pi} \int_{|z| \leq M} \frac{\sin Nz}{z} dz. \end{aligned}$$

By Riemann's lemma, we know  $\exists N_1 > 0$  s.t. when  $N \geq N_1$ , we have

$$\frac{\|f'\|_{L^\infty}}{\pi} \int_{|z| \leq M} \sin Nz dz \leq \frac{\varepsilon}{4}.$$

□

### Fourier transform for $L^2$ -functions

**Theorem 3.3.** *If  $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ , then  $\hat{f} \in L^2(\mathbb{R})$  and*

$$\|\hat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}.$$

Furthermore,  $f \rightarrow \hat{f}$  has a unique extension to a continuous, linear map from  $L^2(\mathbb{R})$  to  $L^2(\mathbb{R})$ , which is isometry.

*Proof.* Due to the fact that  $C_0^\infty$  is dense in  $L^p$ , for  $p \geq 1$ . We only need to prove the equation for all  $f \in L^\infty$ .  $\forall \varepsilon > 0$ , we consider the following weighted integral

$$\int_{\mathbb{R}} |\hat{f}(k)|^2 e^{-\varepsilon|k|^2} dk.$$

By the definition of Fourier transform, we have

$$\int_{\mathbb{R}} |\hat{f}(k)|^2 e^{-\varepsilon|k|^2} dk = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{f(x)} f(y) e^{ik(x-y)} e^{-\varepsilon|k|^2} dx dy dk.$$

Due to the fact that  $\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\varepsilon k^2} e^{ik(x-y)} dk = (e^{-\varepsilon k^2})^\vee(x-y)$ , with the help of Fubini theorem, the above integral is exactly

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\varepsilon}} e^{-\frac{(x-y)^2}{4\varepsilon}} \overline{f(x)} f(y) dx dy.$$

Since

$$\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\left(\frac{x}{2\sqrt{\varepsilon}}\right)^2} d\left(\frac{x}{2\sqrt{\varepsilon}}\right) = 1,$$

It is easy to check that for  $f \in C_0^\infty$ ,

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\varepsilon}} e^{-\frac{(x-y)^2}{4\varepsilon}} f(y) dy \rightarrow f(x), \quad \text{for all } x \in \mathbb{R}.$$

Thus

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} |\hat{f}(k)|^2 e^{-\varepsilon|k|^2} dk = \int_{\mathbb{R}} |f|^2 dx.$$

Then monotone convergence theorem implies that  $\hat{f} \in L^2$  and

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2}.$$

If  $f \in L^2$  but not in  $C_0^\infty$ , since  $C_0^\infty$  is dense in  $L^2$ , there exists  $\{f_j\} \subset C_0^\infty$  such that

$$\|f_j - f\|_{L^2} \rightarrow 0.$$

On the other hand, since Fourier transform is linear, we have

$$\|\hat{f}_j - \hat{f}_m\|_{L^2} = \|f_j - f_m\|_{L^2} \rightarrow 0, \quad \text{as } j, m \rightarrow \infty.$$

Hence,  $\{\hat{f}_j\}$  is a Cauchy sequence in  $L^2$ . Since  $L^2$  is a complete space, we have  $\exists g \in L^2$  such that  $\hat{f}_j \rightarrow g$  strongly in  $L^2$ .

Now we define  $\hat{f} = g$ , thus we have

$$\|\hat{f}\|_{L^2} = \lim_{j \rightarrow \infty} \|\hat{f}_j\|_{L^2} = \lim_{j \rightarrow \infty} \|f_j\|_{L^2} = \|f\|_{L^2}.$$

Continuity and linearity are left to readers.  $\square$

*Remark 3.2.* Fourier transform can be extended for  $L^p$  functions in a similar way by using the following inequality

$$\|\hat{f}\|_{L^q} \leq C(p, q) \|f\|_{L^p}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

3.1.2. *Distribution and weak derivative.* Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ .

## Distributions

**Definition 3.** Test function space  $\mathcal{D}(\Omega)$  consists of all the functions in  $C_0^\infty(\Omega)$  supplemented by the following convergence:  $\phi_m \rightarrow \phi \in C_0^\infty(\Omega)$  iff

- (1)  $\exists$  a compact set  $K \subset \Omega$  such that  $\text{supp} \phi_m \subset K, \forall m,$

(2) for all multi-index  $\forall \alpha$ ,

$$\sup_K |\partial^\alpha \phi_m - \partial^\alpha \phi| \rightarrow 0, \text{ as } m \rightarrow \infty.$$

*Remark 3.3.*  $\mathcal{D}(\Omega)$  is a linear space.

**Definition 4.** Distribution, denoted by  $\mathcal{D}'(\Omega)$ , is the dual space of  $\mathcal{D}(\Omega)$ , i.e., is the linear space that includes all the continuous linear functionals on  $\mathcal{D}(\Omega)$ .  $T : \mathcal{D}(\Omega) \rightarrow \mathbb{C}$  is called continuous linear iff

- (1)  $\langle T, \alpha\phi_1 + \beta\phi_2 \rangle = \alpha\langle T, \phi_1 \rangle + \beta\langle T, \phi_2 \rangle$ ,
- (2) If  $\phi_m \rightarrow \phi$  in  $\mathcal{D}(\Omega)$ , then  $\langle T, \phi_m \rangle \rightarrow \langle T, \phi \rangle$ .

*Remark 3.4.* It is usually nonsense to multiply two distributions, since it is not well defined.

*Remark 3.5.* A distribution multiplied by a smooth function can be defined by the following. For  $T \in \mathcal{D}'$  and  $f \in C^\infty$ , their product is defined by

$$\langle Tf, \phi \rangle = \langle T, f\phi \rangle, \quad \forall \phi \in \mathcal{D}.$$

*Remark 3.6.* The support of a distribution and the convolution of two distributions can be also defined with the help of test functions, since we will not use these in our lecture, we omit the detail here.

**Example 2.**  $L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$ .

$\forall f \in L^1_{loc}(\Omega)$ ,  $T_f \in \mathcal{D}'(\Omega)$  is defined by

$$\langle T_f, \phi \rangle = \int_{\Omega} f(x)\phi(x)dx, \quad \forall \phi \in \mathcal{D}(\Omega).$$

*Remark 3.7.* Similarly,  $L^p_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$ . One can use the Hölder's inequality to obtain that  $L^p_{loc}(\Omega) \subset L^q_{loc}(\Omega)$  for  $1 < q < p < \infty$ .

**Theorem 3.4.**  $L^1_{loc}$  functions are uniquely determined by distributions. More precisely, for two functions  $f, g \in L^1_{loc}(\Omega)$ , if

$$\int_{\Omega} f\phi dx = \int_{\Omega} g\phi dx, \quad \forall \phi \in \mathcal{D}(\Omega),$$

then  $f = g$  a.e. in  $\Omega$ .

The proof is left to the readers.

**Example 3.** Probability density functions on  $\mathbb{R}$  is a subset of  $\mathcal{D}'(\mathbb{R})$ . For any probability density function  $P(x)$ ,  $T_P \in \mathcal{D}'(\mathbb{R})$  is defined by

$$\langle T_P, \phi \rangle = \int_{\mathbb{R}} \phi(x)P(x)dx, \quad \forall \phi \in \mathcal{D}(\mathbb{R}).$$

**Example 4.** The set of measures,  $\mathcal{M}(\Omega)$ , is a subset of  $\mathcal{D}'(\Omega)$ . For any  $\mu \in \mathcal{M}(\Omega)$ ,  $T_\mu \in \mathcal{D}'(\Omega)$  is defined by

$$\langle T_\mu, \phi \rangle = \int_{\Omega} \phi(x)d\mu, \quad \forall \phi \in \mathcal{D}(\Omega).$$

**Example 5.** Dirac delta function.

As a typical example of measure, Delta function  $\delta(x)$  is defined by

$$\langle \delta, \phi \rangle = \phi(0), \quad \forall \phi \in \mathcal{D}(\Omega).$$

Remark 3.8.  $\delta \notin L^1_{loc}$ .

*Proof.* If not, there exists  $f \in L^1_{loc}$  s.t.  $\forall \phi \in C^\infty_0$ ,

$$\langle \delta, \phi \rangle = \int_{\mathbb{R}} f \phi dx.$$

Now we choose  $\varphi_n(x) = \begin{cases} e^{\frac{1}{|nx|^{2-1}}}, & |nx| < 1 \\ 0, & |nx| \geq 1 \end{cases}$ . Then we have on one hand

$$\langle \delta, \phi_n \rangle = \phi_n(0) = e^{-1}.$$

On the other hand, since  $f \in L^1_{loc}$ ,

$$\int_{\mathbb{R}} f \varphi_n dx = \int_{|x| \leq \frac{1}{n}} f(x) e^{\frac{1}{|nx|^{2-1}}} dx \rightarrow 0, \text{ as } n \rightarrow \infty,$$

which is a contradiction. □

**Definition 5.** A sequence of distributions  $\{T_n\}_{n=1}^\infty$  converges to  $T$  in the sense of distribution if

$$\langle T_n, \phi \rangle \rightarrow \langle T, \phi \rangle, \quad \forall \phi \in \mathcal{D}(\Omega).$$

In the following we will show some sequences which converge to  $\delta$ -function in the sense of distribution.

**Example 6.** Heat kernel  $f_t(x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$ .

$$\int_{\mathbb{R}} f_t(x) dx = 1 \text{ and } \forall \phi \in C^\infty_0,$$

$$\int_{\mathbb{R}} f_t(x) \phi(x) dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} \phi(2\sqrt{t}y) dy \rightarrow \phi(0),$$

by dominated convergence.

**Example 7.**  $Q_n(x) = \begin{cases} \frac{n}{2}, & |nx| < 1 \\ 0, & |nx| \geq 1 \end{cases}$ .

$$\int_{\mathbb{R}} Q_n(x) dx = 1 \text{ and } \forall \phi \in C^\infty_0,$$

$$\int_{\mathbb{R}} Q_n(x) \phi(x) dx = \int_{-\frac{1}{n}}^{\frac{1}{n}} \frac{n}{2} \phi(x) dx \rightarrow \phi(0).$$

**Example 8.** Dirichlet kernel  $D_n(x) = \frac{\sin(n + \frac{1}{2})x}{\sin \frac{x}{2}} = 1 + 2 \sum_{k=1}^n \cos kx$ .

$$\int_{\mathbb{R}} D_n(x) dx = 2\pi \text{ and } \forall \phi \in C_0^\infty,$$

$$\int_{-\pi}^{\pi} D_n(x) \phi(x) dx = \int_{-\pi}^{\pi} \frac{\sin(n + \frac{1}{2})x}{\sin \frac{x}{2}} \phi(x) dx \rightarrow 2\pi \phi(0),$$

which can be proved by using Riemann's lemma and the similar argument to those we have used in the proof of inverse Fourier transform for  $L^1 \cap C^1$  functions.

**Weak derivative of distributions** The definition of weak derivative is enlightened by integration by parts. If  $f \in C^1$ , we have

$$\int_{\Omega} \partial_i f \phi dx = - \int_{\Omega} f \partial_i \phi dx, \quad \forall \phi \in C_0^\infty.$$

With the same idea, we can define the weak derivative for distributions.

**Definition 6.**  $\forall T \in \mathcal{D}'(\Omega)$ ,  $\partial_i T$  is defined by

$$\langle \partial_i T, \phi \rangle := -\langle T, \partial_i \phi \rangle, \quad \forall \phi \in \mathcal{D}(\Omega).$$

Since  $-\partial_i \phi \in \mathcal{D}'(\Omega)$ , it is easy to check that the right hand side defines a continuous linear functional on  $\mathcal{D}'(\Omega)$ . This new linear functional, as a distribution, is called the weak derivative of  $T$ . One can define higher order derivatives in the same way. Let  $\alpha$  be a multi-index,

$$\langle \partial^\alpha T, \phi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \phi \rangle, \quad \forall \phi \in \mathcal{D}(\Omega).$$

*Remark 3.9.* According to this definition, we know that all distributions are infinitely weakly differentiable.

**Example 9.** *The derivatives of  $\delta$ -function.*

$\forall \phi \in \mathcal{D}(\mathbb{R})$ , we have

$$\begin{aligned} \langle \delta', \phi \rangle &= -\langle \delta, \phi' \rangle = -\phi'(0), \\ \langle \delta^{(k)}, \phi \rangle &= (-1)^k \langle \delta, \phi^{(k)} \rangle = (-1)^k \phi^{(k)}(0). \end{aligned}$$

**Example 10.** *The derivatives of Heaviside function  $H = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ .*

$\forall \phi \in \mathcal{D}(\mathbb{R})$ , we have

$$\langle H', \phi \rangle = -\langle H, \phi' \rangle = - \int_0^\infty \phi'(x) dx = \phi(0) = \langle \delta, \phi \rangle.$$

Motivated by the change of variables, we can give the translation of distributions.

**Definition 7.**

$$\langle T(x - a), \phi(x) \rangle = \langle T(x), \phi(x + a) \rangle.$$

For example,  $\delta_a(x) = \delta(x - a)$  is defined by

$$\langle \delta, \phi(x + a) \rangle = \phi(a).$$

3.1.3. *Tempered distribution and its Fourier transform.*

**Definition 8.** Schwartz class function  $\mathcal{S}(\mathbb{R}^n)$ , (rapidly decreasing function),

$$\mathcal{S}(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) \mid \sup_x |x^\gamma D^\alpha \phi| < +\infty, \forall \text{ multi-index } \alpha, \gamma\},$$

where the convergence of a sequence  $\{\phi_j\} \subset \mathcal{S}(\mathbb{R}^n)$  to  $\phi \in \mathcal{S}(\mathbb{R}^n)$  is defined by

$$\sup_x |x^\gamma D^\alpha (\phi_j - \phi)| \rightarrow 0, \forall \text{ multi-index } \alpha, \gamma.$$

**Theorem 3.5.** *If  $\phi \in \mathcal{S}(\mathbb{R}^n)$ , then  $\hat{\phi} \in \mathcal{S}(\mathbb{R}^n)$*

*Proof.* For all multi-index  $\alpha, \gamma$ , by the properties of Fourier transform, we have

$$\left| k^\alpha D^\gamma \hat{\phi}(k) \right| = \left| \left( D^\alpha ((-x)^\gamma \phi(x)) \right)^\wedge \right| \leq \int_{\mathbb{R}^n} |D^\alpha ((-x)^\gamma \phi(x))| dx.$$

By taking the supremum in  $k \in \mathbb{R}^n$ , we have

$$\begin{aligned} \sup_k |k^\alpha D^\gamma \hat{\phi}(k)| &\leq \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |D^\alpha ((-x)^\gamma \phi(x))| dx \\ &\leq \frac{1}{(2\pi)^{\frac{n}{2}}} C \sup_x (1 + |x|)^{n+1} |D^\alpha ((-x)^\gamma \phi(x))| < +\infty, \end{aligned}$$

where  $C = \int_{\mathbb{R}^n} \frac{1}{(1 + |x|)^{n+1}} dx.$

□

**Definition 9.** The dual space of  $\mathcal{S}(\mathbb{R}^n)$  is called tempered distribution space, denoted by  $\mathcal{S}'(\mathbb{R}^n)$ .

*Remark 3.10.*

$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n), \quad \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n).$$

**Definition 10.**  $\forall T \in \mathcal{S}'(\mathbb{R}^n)$ , its Fourier transform is defined by  $\forall \phi \in \mathcal{S}(\mathbb{R}^n)$ ,

$$\langle \hat{T}, \phi \rangle = \langle T, \hat{\phi} \rangle.$$

**Example 11.** *The Fourier transform of Dirac delta function is a constant.*

$\forall \phi \in \mathcal{S}(\mathbb{R}^n)$ , by definition,

$$\langle \hat{\delta}, \phi \rangle = \langle \delta, \hat{\phi} \rangle = \hat{\phi}(0) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \phi(x) e^{-i0 \cdot x} dx = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \phi(x) dx = \left\langle \frac{1}{(2\pi)^{\frac{n}{2}}}, \phi \right\rangle$$

This means that  $\hat{\delta} = \frac{1}{(2\pi)^{\frac{n}{2}}}$  in the sense of distributions.

**3.2. Cauchy Problem.** The initial value problem of the heat equation is

$$u_t - \Delta u = f(x, t), \quad x \in \mathbb{R}^n, t > 0, \tag{3.1}$$

$$u|_{t=0} = u_0(x). \tag{3.2}$$

3.2.1. *Solution formula of the Cauchy problem.* We will find the formal solution of Cauchy problem by Fourier transform. Taking the Fourier transform in  $x$  in equation (3.1) and as well as its initial value in (3.2) gives

$$\begin{aligned}\hat{u}_t + |k|^2 \hat{u} &= \hat{f}(k, t), \quad k \in \mathbb{R}^n, t > 0, \\ \hat{u}|_{t=0} &= \hat{u}_0(k).\end{aligned}$$

This ODE problem is easy to solve by Duhamel's formula

$$\hat{u}(k, t) = e^{-|k|^2 t} \hat{u}_0(k) + \int_0^t e^{-|k|^2 (t-\tau)} \hat{f}(k, \tau) d\tau.$$

Now taking the inverse Fourier transform and using its property for convolutions, we have

$$\begin{aligned}u(x, t) &= \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} * u_0(x) + \int_0^t \frac{1}{(4\pi(t-\tau))^{\frac{n}{2}}} e^{-\frac{|x|^2}{4(t-\tau)}} * f(x, \tau) d\tau \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy + \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-\tau))^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(t-\tau)}} f(y, \tau) dy d\tau.\end{aligned}$$

It can be seen from here that the function, known as **heat kernel**,

$$K(x, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, \quad (3.3)$$

is very important in getting the solution of the heat equation. Actually, the solution can be written in the following form

$$u(x, t) = K(x, t) * u_0 + \int_0^t K(x, t-\tau) * f(x, \tau) d\tau. \quad (3.4)$$

which is called **Poisson's formula**.

**Theorem 3.6.** *For homogeneous equation, i.e.,  $f \equiv 0$ , if  $u_0$  is a bounded function in  $C(\mathbb{R})$ , then  $u(x, t)$  given by (3.4) is a bounded classical solution of (3.1)-(3.2).*

*Proof.* It is easy to see that  $\forall t > 0$ ,  $u(x, t) = K(x, t) * u_0(x)$  is infinitely differentiable both in  $x$  and  $t$ . Another fact is that

$$K_t - \Delta K = 0, \quad \forall t > 0.$$

These show that  $u(x, t)$  is a solution of the equation for  $t > 0$ , i.e.,  $u_t - \Delta u = 0$  in  $\mathbb{R}^n \times (0, \infty)$ . Now we are left to check whether  $u(x, t)$  satisfies the initial data. More precisely, it remains to show that,

$$\lim_{(x,t) \rightarrow (x_0,0)} u(x, t) = u_0(x_0), \quad \forall x_0 \in \mathbb{R}^n.$$

By changing of variables in the integral, we have

$$\begin{aligned}u(x, t) &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy \\ &= \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-|z|^2} u_0(x + 2\sqrt{t}z) dz.\end{aligned}$$

Due to the fact that  $u_0$  is bounded, the above integral is uniformly convergent in  $x$  and  $t$ , which allows us to switch the order of the limit and the integral. Therefore

$$\lim_{(x,t) \rightarrow (x_0,0)} u(x,t) = \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-|z|^2} u_0(x_0) dz = u_0(x_0).$$

□

*Remark 3.11.* Some basic properties of the solution  $u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy$  can be obtained directly from this formula.

- (1) If  $u_0$  is periodic (or odd, or even) in  $x$ , so is  $u(x,t)$ .
- (2) Infinite speed of propagation. If  $u_0(x) \geq 0$  has support in a small domain, say  $\Omega_0 \subset \mathbb{R}^n$ ,  $u(x,t)$  is positive everywhere in  $\mathbb{R}^n$ .
- (3) Infinite differentiability of the solution  $u(x,t)$  for  $t > 0$ .

Next we consider the inhomogeneous equation with homogeneous initial value

$$\begin{aligned} u_t - \Delta u &= f, \quad \text{in } \mathbb{R}^n \times (0, +\infty), \\ u|_{t=0} &= 0. \end{aligned} \tag{3.5}$$

The solution is given by

$$\begin{aligned} u(x,t) &= \int_0^t \int_{\mathbb{R}^n} K(x-y, t-s) f(y,s) dy ds \\ &= \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s) dy ds. \end{aligned} \tag{3.6}$$

**Theorem 3.7.** *If  $f \in C^{2,1}(\mathbb{R}^n \times [0, \infty))$  and has compact support, then the function  $u$  given by (3.6) is a function in  $C^{2,1}(\mathbb{R}^n \times [0, \infty))$ , furthermore, it is a solution of (3.5).*

*Proof.* By the regularity of  $f$ , we have

$$\begin{aligned} &(\partial_t - \Delta)u(x,t) \\ &= \int_0^t \int_{\mathbb{R}^n} K(y,s) (f_t - \Delta f)(x-y, t-s) dy ds + \int_{\mathbb{R}^n} K(y,t) f(x-y,0) dy \\ &= \int_0^\varepsilon \int_{\mathbb{R}^n} K(y,s) (f_t - \Delta f)(x-y, t-s) \\ &\quad + \int_\varepsilon^t \int_{\mathbb{R}^n} K(y,s) (f_t - \Delta f)(x-y, t-s) \\ &\quad + \int_{\mathbb{R}^n} K(y,t) f(x-y,0) dy =: J_\varepsilon + I_\varepsilon + L. \end{aligned}$$

We deal with the right hand side term by term,

$$|J_\varepsilon| \leq (\|f_t\|_{L^\infty} + \|D^2 f\|_{L^\infty}) \int_0^\varepsilon \int_{\mathbb{R}^n} K(y,s) dy ds \leq \varepsilon C.$$

$$\begin{aligned}
I_\varepsilon &= \int_\varepsilon^t \int_{\mathbb{R}^n} K(y, s)(f_t - \Delta f)(x - y, t - s) dy ds \\
&= \int_\varepsilon^t \int_{\mathbb{R}^n} (\partial_s - \Delta_y) K(y, s) f(x - y, t - s) dy ds + \int_{\mathbb{R}^n} K(y, \varepsilon) f(x - y, t - \varepsilon) dy \\
&\quad - \int_{\mathbb{R}^n} K(y, t) f(x - y, 0) dy \\
&= \int_{\mathbb{R}^n} K(y, \varepsilon) f(x - y, t - \varepsilon) dy - L.
\end{aligned}$$

Therefore we have

$$u_t - \Delta u = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} K(y, \varepsilon) f(x - y, t - \varepsilon) dy = f(x, t), \quad \forall t > 0,$$

and

$$|u(x, t)| = \left| \int_0^t \int_{\mathbb{R}^n} K(y, s) f(x - y, t - s) dy ds \right| \leq t \|f\|_{L^\infty} \rightarrow 0, \quad \text{as } t \rightarrow 0,$$

which means that the solution also satisfies the initial condition.  $\square$

*Remark 3.12.* By superposition principle for linear equations, under the conditions of theorem 3.6 and 3.7, we claim

$$u(x, t) = \int_{\mathbb{R}^n} K(x - y, t) u_0(y) dy + \int_0^t \int_{\mathbb{R}^n} K(x - y, t - s) f(y, s) dy ds$$

is the solution of

$$u_t - \Delta u = f, \quad u|_{t=0} = u_0.$$

**3.2.2. Fundamental solution.** We first give some motivations in defining fundamental solutions. Formally, the right hand side function  $f(x, t)$ , the heat source, can be represented by

$$f(x, t) = \int_0^\infty \int_{\mathbb{R}^n} \delta_{(\xi, \tau)}(x, t) f(\xi, \tau) d\xi d\tau, \quad \forall x \in \mathbb{R}^n, t > 0,$$

which means that  $f(x, t)$  can be treated as a summation of  $\delta_{(\xi, \tau)}(x, t) f(\xi, \tau) d\xi d\tau$ , the point heat source. Therefore, we can expect that if  $K(x, t; \xi, \tau)$  is the solution of  $u_t - \Delta_x u = \delta_{(\xi, \tau)}(x, t)$ , then  $K(x - \xi, t - \tau) f(\xi, \tau) d\xi d\tau$  is the solution with point heat source. Thus in the case of general heat source  $f(x, t)$ , the solution can be written into a superposition of point heat sources,

$$u(x, t) = \int_0^\infty \int_{\mathbb{R}^n} K(x - \xi, t - \tau) f(\xi, \tau) d\xi d\tau.$$

Basically, the fundamental solution of heat equation is to find the temperature distribution with a point heat source at  $(\xi, \tau)$ .

**Definition 11.**  $K(x, t; \xi, \tau) = K(x - \xi, t - \tau)$  is called the **fundamental solution** of heat equation.

Let  $Q = \mathbb{R}^n \times (0, \infty)$ .  $\forall(x, t) \in Q$ .  $K(x, t; \xi, \tau)$  is a solution (in the sense of distribution) of the following Cauchy problem

$$\begin{aligned} u_t - \Delta_x u &= \delta_{(\xi, \tau)}(x, t), \\ u|_{t=0} &= 0. \end{aligned}$$

*Remark 3.13.* We also know that  $K(x, t; \xi, \tau)$  is a solution of

$$\begin{aligned} u_t - \Delta_x u &= 0, \quad \forall x \in \mathbb{R}^n, t \geq \tau, \\ u|_{t=\tau} &= \delta_\xi(x). \end{aligned}$$

### Some properties of the fundamental solution

- (1)  $K(x, t; \xi, \tau) > 0$  for  $t > \tau$ .
- (2)  $K(x, t; \xi, \tau) = K(\xi, t; x, \tau)$ .
- (3)  $\forall x \in \mathbb{R}^n, t > \tau$ ,

$$\int_{\mathbb{R}^n} K(x, t; \xi, \tau) d\xi = 1.$$

- (4)  $\forall x, \xi \in \mathbb{R}^n, t > \tau$ ,

$$\begin{aligned} (\partial_t - \Delta_x)K(x, t; \xi, \tau) &= 0 \\ (\partial_\tau + \Delta_\xi)K(x, t; \xi, \tau) &= 0 \end{aligned}$$

- (5) If  $\varphi(x)$  is a bounded continuous function in  $\mathbb{R}^n$ , then

$$\lim_{t \rightarrow 0^+} \int_{\mathbb{R}^n} K(x, t; \xi, 0) \varphi(\xi) d\xi = \varphi(x).$$

- (6)  $K(x, t; \xi, \tau)$  is infinitely differentiable and  $\exists M > 0$  s.t. in the case of  $t > \tau$ ,

$$|K(x, t; \xi, \tau)| \leq \frac{M}{(t - \tau)^{\frac{n}{2}}}.$$

*Remark 3.14.* There is another derivation of fundamental solution instead of using Fourier transform. One can check this method in Evan's book.

**3.3. Half space problem and its Green's function.** The main purpose of this section is to give a first insight on how to build up a Green's function on general problem.

Consider the problem

$$\begin{aligned} u_t - u_{xx} &= 0, \quad x \in (0, +\infty), t > 0, \\ u|_{t=0} &= \varphi, \quad x \in [0, +\infty), \\ u|_{x=0} &= 0, \quad t > 0. \end{aligned} \tag{3.7}$$

We want to find a function  $G(x, t, \xi, 0)$  such that the solution of (3.7) can be represented by

$$u(x, t) = \int_0^\infty G(x, t; \xi, 0) \varphi(\xi) d\xi.$$

The important thing here is that we must make sure the solution satisfies the boundary condition  $u|_{x=0} = 0$ .  $\forall \xi \in (0, +\infty)$ , if the initial data is  $\delta_\xi(x)$ , we need to find the odd extension of it, i.e.,  $-\delta_{-\xi}(x)$ , to balance the boundary condition. Now we can choose the initial data as

$$\delta_\xi(x) - \delta_{-\xi}(x),$$

and solve the Cauchy problem with this initial data. Since the problem is linear, our the solution is exactly

$$K(x, t; \xi, 0) - K(x, t; -\xi, 0).$$

Thus the Green's function for half space problem (3.7) can be written as

$$G(x, t; \xi, 0) = K(x, t; \xi, 0) - K(x, t; -\xi, 0),$$

and the solution of (3.7) is expected to be  $u(x, t) = \int_0^\infty G(x, t; \xi, 0)\varphi(\xi)d\xi$ .

**Theorem 3.8.**  $\varphi$  is a bounded smooth function on  $(0, +\infty)$  and  $\varphi(0) = 0$ ,  $u(x, t) = \int_0^\infty G(x, t; \xi, 0)\varphi(\xi)d\xi$  is the solution of (3.7).

The proof of this theorem is left to the reader.

*Remark 3.15.* For inhomogeneous problem

$$\begin{aligned} u_t - u_{xx} &= f, & x \in (0, +\infty), t > 0, \\ u|_{t=0} &= \varphi, & x \in [0, +\infty), \\ u|_{x=0} &= 0, & t > 0. \end{aligned}$$

The formal solution is

$$u(x, t) = \int_0^\infty G(x, t; \xi, 0)\varphi(\xi)d\xi + \int_0^t d\tau \int_0^\infty G(x, t; \xi, \tau)f(\xi, \tau)d\xi.$$

*Remark 3.16.* Similarly, one can find the Green's function for half space problem with homogeneous Neumann boundary condition.

**3.4. Initial boundary value problem.** The initial boundary value (with homogeneous Dirichlet boundary condition) problem for heat equation in 1-d space variable is

$$\begin{aligned} u_t - u_{xx} &= f, & x \in (0, 1), t > 0, \\ u|_{t=0} &= \varphi, & x \in (0, 1), \\ u|_{x=0} = u|_{x=1} &= 0, & t > 0. \end{aligned} \tag{3.8}$$

The method of separation of variables is easy to be applied here.

3.4.1. *Separation of variables.* First by solving the eigenvalue problem

$$\begin{aligned} X'' + \lambda X &= 0, & x \in (0, 1), \\ X(0) &= X(1) = 0, \end{aligned}$$

we have

$$\lambda_n = (n\pi)^2, \quad X_n = \sin n\pi x.$$

Then if the solution  $u(x, t)$  has form

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin n\pi x,$$

$T_n(t)$  will solve the initial value problem of an ODE

$$\begin{aligned} T_n' + (n\pi)^2 T_n &= f_n(t), \\ T_n(0) &= \varphi_n, \end{aligned}$$

where

$$f_n(t) = 2 \int_0^1 f(x, t) \sin n\pi x dx, \quad \varphi_n = 2 \int_0^1 \varphi(x) \sin n\pi x dx.$$

This ODE problem has a solution

$$T_n(t) = e^{-(n\pi)^2 t} \varphi_n + \int_0^t e^{-(n\pi)^2 (t-\tau)} f_n(\tau) d\tau, \quad n = 1, 2, \dots$$

Thus the formal solution for problem (3.8) can be written as

$$u(x, t) = \sum_{n=1}^{\infty} \sin n\pi x \left( e^{-(n\pi)^2 t} \varphi_n + \int_0^t e^{-(n\pi)^2 (t-\tau)} f_n(\tau) d\tau \right). \quad (3.9)$$

A natural question to ask is under what condition is (3.9) a  $C^{2,1}((0, 1) \times (0, \infty))$  solution. Here we give a result with sufficient assumption on the given data.

**Theorem 3.9.** *If  $\varphi(x) \in C(0, 1)$  and  $f, f_t, f_{xxt} \in C_0^{2,1}((0, 1) \times (0, \infty))$ , then the formula given by (3.9) is in  $C^{2,1}$  and is a solution of the problem (3.8).*

*Proof.* The uniform convergence of the formula  $u(x, t)$  is omitted here. The more difficult steps are shown in the following, i.e., to prove that  $u_{xx}$  and  $u_t$  also converge uniformly in  $x, t$ .

If we take twice derivatives in  $x$  or derivative in  $t$  in the formula (3.9) on each term in the summation, we will have  $(n\pi)^2$ . Therefore, in order to prove that it is the solution, we only need to show that  $\forall t > 0$ , the following series converge uniformly in  $x$  and  $t$ , i.e.,

$$\begin{aligned} & \sum_{n=1}^{\infty} (n\pi)^2 \sin n\pi x e^{-(n\pi)^2 t} \varphi_n; \\ & \sum_{n=1}^{\infty} (n\pi)^2 \sin n\pi x \int_0^t e^{-(n\pi)^2 (t-\tau)} f_n(\tau) d\tau. \end{aligned}$$

The convergence of the first series is obvious because of the exponential term  $e^{-(n\pi)^2 t}$ . To deal with the second series, we need to “produce” a factor  $\frac{1}{(n\pi)^2}$  by using integration by parts on the integrals involved in the formula.

$$\begin{aligned} & \int_0^t e^{-(n\pi)^2(t-\tau)} f_n(\tau) d\tau \\ &= \frac{f_n(t)}{(n\pi)^2} - \frac{e^{-(n\pi)^2 t} f_n(0)}{(n\pi)^2} - \int_0^t \frac{1}{(n\pi)^2} e^{-(n\pi)^2(t-\tau)} f_n'(\tau) d\tau. \end{aligned}$$

Therefore, the second series can be rewritten into

$$\begin{aligned} & \sum_{n=1}^{\infty} (n\pi)^2 \sin n\pi x \int_0^t e^{-(n\pi)^2(t-\tau)} f_n(\tau) d\tau \\ &= \sum_{n=1}^{\infty} \sin n\pi x \left( f_n(t) - e^{-(n\pi)^2 t} f_n(0) - \int_0^t e^{-(n\pi)^2(t-\tau)} f_n'(\tau) d\tau \right) \\ &= f(x, t) - f_n(0) \sum_{n=1}^{\infty} \sin n\pi x e^{-(n\pi)^2 t} - \sum_{n=1}^{\infty} \sin n\pi x \int_0^t e^{-(n\pi)^2(t-\tau)} f_n'(\tau) d\tau, \end{aligned}$$

where the convergence of the last series can be done by using exactly the same method as for the convergence of  $u(x, t)$  itself. Namely, with the assumption that  $f$  has compact support, we do integration by parts twice,

$$\begin{aligned} f_n'(\tau) &= 2 \int_0^1 f_t(x, \tau) \sin n\pi x dx \\ &= -\frac{2}{(n\pi)^2} \int_0^1 f_{xxt}(x, \tau) \sin n\pi x dx =: -\frac{2B_n(\tau)}{(n\pi)^2}. \end{aligned}$$

Due to the fact that  $\sum_{n=1}^{\infty} B_n^2 < \infty$ , we have

$$\begin{aligned} & \left| -\sum_{n=1}^{\infty} \sin n\pi x \int_0^t e^{-(n\pi)^2(t-\tau)} f_n'(\tau) d\tau \right| \\ & \leq \sum_{n=1}^{\infty} \frac{2|\sin n\pi x|}{(n\pi)^2} \int_0^t e^{-(n\pi)^2(t-\tau)} |B_n(\tau)| d\tau \\ & \leq \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2} \int_0^t (1 + B_n^2(\tau)) d\tau < \infty. \end{aligned}$$

□

**Basic properties of the solution of heat (or more generally, parabolic) equation.** In the case of no external source, i.e.,  $f = 0$ , the solution has “Infinitely differentiability inside of the domain”. It is mainly due to that  $e^{-(n\pi)^2 t}$  decays exponentially in  $n$  for any positive time  $t$ . More precisely,  $\forall (x, t) \in (0, 1) \times (0, \infty)$ , for any nonnegative integer

$k, l$ , the solution given by (3.9) is  $(k + l)$  times differentiable at  $(x, t)$ . For example, we know that

$$\frac{\partial^{k+l}u(x, t)}{\partial x^k \partial t^l} = \sum_{n=1}^{\infty} (-1)^l (n\pi)^{k+2l} \varphi_n e^{-(n\pi)^2 t} \sin(n\pi x + \frac{k\pi}{2})$$

converges uniformly in  $x$  and positive  $t$ .

**3.4.2. Energy estimates.** We will give the energy estimate for initial boundary value problem of heat equation in multi-dimension.  $\Omega$  is a bounded open subset of  $\mathbb{R}^n$ . Let  $Q_T = \Omega \times (0, T)$ .

$$\begin{aligned} u_t - \Delta u &= f, & (x, t) \in Q_T, \\ u|_{t=0} &= \varphi, & x \in \Omega, \\ u|_{\partial\Omega} &= 0, & t > 0. \end{aligned} \tag{3.10}$$

**Theorem 3.10.** *If  $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$  is a solution of problem (3.10), then  $\exists M > 0$  depends only on  $T$ , s.t.*

$$\sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(Q_T)} \leq M(\|\varphi\|_{L^2(\Omega)} + \|f\|_{L^2(Q_T)}). \tag{3.11}$$

*Proof.* By multiplying the equation by  $u$ , integrating it over  $Q_t$  and integrating by parts in  $x$ , we have

$$\frac{1}{2} \int_{\Omega} u^2 dx - \frac{1}{2} \int_{\Omega} \varphi^2 dx + \int_0^t \int_{\Omega} |\nabla u|^2 dx dt = \int_0^t \int_{\Omega} f u dx dt.$$

Young's inequality gives

$$\int_{\Omega} u^2 dx + 2 \int_0^t \int_{\Omega} |\nabla u|^2 dx dt \leq \int_0^t \int_{\Omega} u^2 dx dt + \int_{\Omega} \varphi^2 dx + \int_0^t \int_{\Omega} f^2 dx dt$$

Then (3.11) can be obtained directly from Gronwall's inequality. □

*Remark 3.17.* The discussion on uniqueness and stability of the solution by energy estimates is similar to what we have done for the wave equation.

*Remark 3.18.* For homogeneous Neumann boundary condition  $u \cdot \gamma|_{\partial\Omega} = 0$ , where  $\gamma$  is the unit outer normal vector of  $\partial\Omega$ , the energy estimate can be done similarly.

*Remark 3.19.* For inhomogeneous boundary condition, i.e.,  $u|_{\partial\Omega} = \psi_D$ , one can try to homogenize it or just use  $u - \psi_D$  as test function.

**3.5. Maximum principle.** In this subsection, we introduce one of the basic tools, the so-called maximum principle, for second order elliptic or parabolic equation. It is easy to get the main idea from handling the heat equation.

In the following, we assume that  $\Omega$  is a bounded open subset of  $\mathbb{R}^n$ . Let  $Q_T = \Omega \times (0, T]$ , the parabolic boundary of  $Q_T$  be  $\partial_p Q_T = \Omega \times \{t = 0\} \cup \partial\Omega \times (0, T]$  and  $Lu = u_t - \Delta u$ .

3.5.1. *Weak maximum principle.*

**Theorem 3.11.** *If  $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$  and  $Lu \leq 0$  in  $Q_T$ , then the maximum of  $u$  in  $\bar{Q}_T$  must be achieved on  $\partial_p Q_T$ , i.e.,*

$$\max_{\bar{Q}_T} u(x, t) = \max_{\partial_p Q_T} u. \quad (3.12)$$

*Proof.* We first assume  $Lu < 0$  in  $Q_T$ . If (3.12) is not true, which means  $\exists(x_0, t_0) \in Q_T$  such that

$$u(x_0, t_0) = \max_{\bar{Q}_T} u(x, t),$$

then we know that  $\nabla u(x_0, t_0) = 0$ ,  $\Delta u(x_0, t_0) \leq 0$  and  $u_t(x_0, t_0) \geq 0$ . Therefore,

$$f(x_0, t_0) = Lu(x_0, t_0) \geq 0,$$

which is a contradiction with the assumption  $Lu < 0$ .

If  $Lu \leq 0$ , we introduce an auxiliary function

$$v(x, t) = u(x, t) - \varepsilon t, \quad \forall 0 < \varepsilon \ll 1.$$

Then the application of the heat operator on  $v$  gives

$$Lv = Lu - \varepsilon = f - \varepsilon < 0.$$

By the conclusion we obtained above, we have

$$\max_{\bar{Q}_T} v = \max_{\partial_p Q_T} v.$$

Going back to the variable  $u$ ,

$$\begin{aligned} \max_{\bar{Q}_T} u(x, t) &= \max_{\bar{Q}_T} (v + \varepsilon t) \leq \max_{\bar{Q}_T} v + \varepsilon T \\ &\leq \max_{\partial_p Q_T} v + \varepsilon T = \max_{\partial_p Q_T} (u - \varepsilon t) + \varepsilon T \\ &\leq \max_{\partial_p Q_T} u + \varepsilon T. \end{aligned}$$

In the end, taking  $\varepsilon \rightarrow 0$  finishes the proof, i.e., we have (3.12).  $\square$

By the same discussion or letting  $v = -u$ , we will have

**Corollary 3.1.** *If  $Lu \geq 0$ , then*

$$\min_{\bar{Q}_T} u(x, t) = \min_{\partial_p Q_T} u.$$

A combination of the above two results gives

**Corollary 3.2.** *If  $Lu = 0$ , then both the maximum and the minimum of  $u$  are achieved on the parabolic boundary.*

Now we will have the very useful tool, the **comparison principle**, as a corollary

**Corollary 3.3.** *If  $u, v \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ ,  $Lu \leq Lv$  and  $u|_{\partial_p Q_T} \leq v|_{\partial_p Q_T}$ , then*

$$u(x, t) \leq v(x, t), \quad \text{in } \bar{Q}_T.$$

3.5.2. *Dirichlet boundary value problem.* The initial boundary value problem of heat equation with Dirichlet boundary condition is

$$\begin{aligned} u_t - \Delta u &= f(x, t), & \text{in } Q_T, \\ u|_{t=0} &= \varphi(x), \\ u|_{\partial\Omega} &= g(x, t). \end{aligned} \tag{3.13}$$

**Theorem 3.12.** *If  $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$  is a solution of (3.13), then*

$$\max_{Q_T} |u| \leq FT + B, \tag{3.14}$$

where  $F = \max_{\bar{Q}_T} |f|$ ,  $B = \max\{\max_{\Omega} |\varphi|, \max_{\partial\Omega \times [0, T]} |g|\}$ .

*Proof.* The proof will be obtained by using comparison principle. We introduce auxiliary function  $w(x, t) = Ft + B \pm u(x, t)$ . It is easy to check that

$$\begin{aligned} Lw &= F \pm f \geq 0, \\ w|_{\partial_p Q_T} &\geq Ft + B \pm g|_{\partial_p Q_T} \geq 0. \end{aligned}$$

By comparison principle, Corollary 3.3, we have  $w(x, t) \geq 0$  in  $Q_T$ , which implies

$$|u| \leq FT + B, \quad \text{in } Q_T.$$

□

This maximum estimate can be used to prove the uniqueness and stability of classical solutions.

**Corollary 3.4.**  $C^{2,1}(Q_T) \cap C(\bar{Q}_T)$  solution of (3.13) is unique.

**Corollary 3.5.**  $C^{2,1}(Q_T) \cap C(\bar{Q}_T)$  solution of (3.13) is stable in the following sense. If  $u_1, u_2 \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$  are solutions separately with data  $\varphi_1, f_1, g_1$  and  $\varphi_2, f_2, g_2$ , then

$$\max_{\bar{Q}_T} |u_1 - u_2| \leq \|f_1 - f_2\|_{\infty} T + \|\varphi_1 - \varphi_2\|_{\infty} + \|g_1 - g_2\|_{\infty}.$$

3.6. **Long time behavior of the solution.** In this subsection, we give further discussion of the long time behavior of the solution to heat equation based on energy or entropy estimates.

3.6.1. *Poincaré's inequalities.* First, we introduce Poincaré type of inequalities.  $\Omega$  is a connected open bounded domain in  $\mathbb{R}^n$

**Lemma 3.1.** (*Poincaré's inequality for  $C_0^1(\Omega)$  functions*) For all  $u \in C_0^1(\Omega)$ , it holds

$$\|u\|_{L^2(\Omega)} \leq 2d \|\nabla u\|_{L^2(\Omega)},$$

where  $d = \text{diam } \Omega$ .

*Proof.* For all  $u \in C_0^1(\Omega)$ , without loss of generality we assume

$$\Omega \subset \{x | 0 \leq x_i \leq 2d, 1 \leq i \leq n\} = \bar{Q}.$$

Let  $\tilde{u} = \begin{cases} u, & x \in \bar{\Omega} \\ 0, & x \in \bar{Q} \setminus \bar{\Omega} \end{cases}$ . It is obvious that  $\tilde{u}$  is a piecewise  $C^1$  function, and

$$\tilde{u}|_{\partial\bar{Q}} = 0.$$

By Newton-Leibnitz formula

$$\tilde{u}(x_1, x_2, \dots, x_n) = \int_0^{x_1} \frac{\partial \tilde{u}}{\partial x_1} dx_1,$$

then

$$\tilde{u}^2 = \left( \int_0^{x_1} \frac{\partial \tilde{u}}{\partial x_1} dx_1 \right)^2 \leq x_1 \int_0^{x_1} \left( \frac{\partial \tilde{u}}{\partial x_1} \right)^2 dx_1 \leq 2d \int_0^{2d} \left| \frac{\partial \tilde{u}}{\partial x_1} \right|^2 dx_1.$$

By taking integration over  $Q$ , we have

$$\int_Q \tilde{u}^2 dx \leq 2d \int_Q \int_0^{2d} \left| \frac{\partial \tilde{u}}{\partial x_1} \right|^2 dx_1 dx \leq 4d^2 \int_Q |\nabla \tilde{u}|^2 dx.$$

Therefore,

$$\|u\|_{L^2(\Omega)} \leq 2d \|\nabla u\|_{L^2(\Omega)}.$$

□

**Lemma 3.2.** (*Poincaré's inequality for  $C^1(\Omega)$  functions*) For all  $u \in C^1(\Omega)$ , it holds

$$\|u - \bar{u}\|_{L^2(\Omega)} \leq C(\Omega) \|\nabla u\|_{L^2(\Omega)},$$

where  $\bar{u} = \frac{1}{|\Omega|} \int_{\Omega} u dx$  and  $C$  is a constant depending on  $\Omega$ .

*Proof.* We only prove the case  $\Omega = (0, 1)$ . The more general case can be proved by using Sobolev embedding theorem. Since  $u$  is a continuous function, there exists  $x_0 \in (0, 1)$  such that  $u(x_0) = \bar{u}$ . Then by Newton-Leibnitz formula,

$$(u(x) - \bar{u})^2 = \left( \int_{x_0}^x u'(y) dy \right)^2 \leq (x - x_0) \int_{x_0}^x |u'(y)|^2 dy.$$

Therefore the inequality is obtained by taking integration over  $(0, 1)$ . □

A more general version is given in the following. We will skip the proof here.

**Lemma 3.3.** (*Gaussian Poincaré's inequality for  $C^1(\mathbb{R}^n)$  functions with measure  $d\mu = G dx$* ) For all  $u \in C^1(\mathbb{R}^n)$ , it holds

$$\|u - \bar{u}\|_{L^2(\mathbb{R}^n, d\mu)} \leq \|\nabla u\|_{L^2(\mathbb{R}^n, d\mu)},$$

where  $\bar{u} = \int_{\mathbb{R}^n} u d\mu$  and  $d\mu = G(x) dx = (2\pi)^{-\frac{n}{2}} e^{-\frac{x^2}{2}} dx$ .

3.6.2. *Heat equation with Dirichlet boundary condition.* Next we use these two inequalities to prove the long time behavior of the solution to heat equation.

**Theorem 3.13.** *If  $u \in C^{2,1}(\Omega \times (0, \infty)) \cap C(\bar{\Omega} \times [0, \infty))$  is a solution of the problem (3.10) with  $f = 0$ , then  $\exists C(\Omega)$  such that*

$$\|u(x, t)\|_{L^2(\Omega)} \leq \|\varphi\|_{L^2(\Omega)} e^{-Ct},$$

furthermore for any fixed  $x$ ,  $u(x, t) \rightarrow 0$  as  $t \rightarrow \infty$ .

*Proof.* Use  $u$  as test function in the heat equation and notice that  $u|_{\partial\Omega} = 0$ , we have

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2(x, t) dx + \int_{\Omega} |\nabla u|^2 dx = 0.$$

With the help of lemma 3.1, we have

$$\frac{d}{dt} \int_{\Omega} u^2(x, t) dx + \frac{1}{d} \int_{\Omega} |u|^2 dx \leq 0.$$

Therefore,

$$\int_{\Omega} u^2(x, t) dx \leq e^{-\frac{t}{d}} \int_{\Omega} \varphi^2.$$

□

3.6.3. *Heat equation with Neumann boundary condition\*\*\*.*

**Theorem 3.14.** *If  $u \in C^{2,1}(\Omega \times (0, \infty)) \cap C(\bar{\Omega} \times [0, \infty))$  is a solution of the problem*

$$u_t - \Delta u = 0, \quad \nabla u \cdot \gamma|_{\Omega} = 0, \quad u|_{t=0} = \varphi(x),$$

then  $\exists C(\Omega)$  such that

$$\|u(x, t) - \bar{\varphi}\|_{L^2(\Omega)} \leq \|\varphi - \bar{\varphi}\|_{L^2(\Omega)} e^{-Ct},$$

furthermore for any fixed  $x$ ,  $u(x, t) \rightarrow \bar{\varphi}$  as  $t \rightarrow \infty$ .

*Proof.* A very important fact of the heat equation with homogeneous boundary condition is that

$$\int_{\Omega} u(x, t) dx \equiv \int_{\Omega} \varphi(x) dx,$$

which can be obtained immediately by integrating the equation over  $\Omega$ , where the diffusion term disappears because of the Neumann boundary condition. Thus  $\bar{u} \equiv \bar{\varphi}$ , which is independent of time.

Use  $u - \bar{u}$  as test function in the heat equation and notice that  $u$  satisfies the homogeneous Neumann boundary condition, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} (u(x, t) - \bar{u})^2 dx + \int_{\Omega} |\nabla u|^2 dx = 0.$$

With the help of lemma 3.2, we have

$$\frac{d}{dt} \int_{\Omega} (u(x, t) - \bar{u})^2 dx + \frac{2}{C} \int_{\Omega} (u(x, t) - \bar{u})^2 dx \leq 0.$$

Therefore,

$$\int_{\Omega} (u(x, t) - \bar{u})^2 dx \leq e^{-\frac{2t}{C}} \int_{\Omega} (\varphi - \bar{\varphi})^2.$$

□

3.6.4. *Heat equation, Cauchy problem.* In this part we concentrate on the Cauchy problem of heat equation and by using different scaling of the variable to observe more details of the solution behavior.

Let  $u(y, \tau) \in C^{2,1}$  be the solution of

$$u_{\tau} = \frac{1}{2} \Delta_y u,$$

and with the initial data  $u_0$  such that  $\int u_0 dx = 1$ . Note that the fundamental solution is  $K(y, \tau) = C \frac{1}{\tau^{\frac{n}{2}}} e^{-\frac{y^2}{2\tau}}$ , we do a new space-time rescaling, i.e.,  $t = \log(1 + \tau)$ ,  $x = \frac{y}{(1+\tau)^{\frac{1}{2}}}$  and let

$$u(y, \tau) = v(x, t)(1 + \tau)^{-\frac{n}{2}}$$

to preserve the total mass of the solution, i.e.,  $\int_{\mathbb{R}^n} v(x, t) dx = \int_{\mathbb{R}^n} u(y, t) dy$ .  $v$  satisfies the following so called Fokker-Planck equation

$$v_t = \frac{1}{2} \Delta_x v + \frac{1}{2} \nabla \cdot (xv) = \frac{1}{2} \nabla \cdot (v(\nabla \ln v + x)).$$

It is obvious that  $G = \pi^{-n/2} e^{-\frac{x^2}{2}}$  is a stationary solution of it.

If we take further  $w = \frac{v}{G}$ ,  $w$  satisfies the following so called Ornstein-Uhlenbeck equation (a symmetrically weighted heat equation)

$$w_t = \frac{1}{2} G^{-1} \nabla \cdot (G \nabla w).$$

Notice that all of the transformation keeps the total mass, i.e.,

$$\int_{\mathbb{R}^n} w d\mu = \int_{\mathbb{R}^n} v(x, t) dx = \int_{\mathbb{R}^n} u(y, t) dy = 1.$$

**Theorem 3.15** (Energy estimate). *Let  $u(y, \tau)$  be a solution of the heat equation and initial data satisfies  $\int_{\mathbb{R}^n} |u_0 - G|^2 dx < \infty$ , then there exists a constant  $C$  such that*

$$\int_{\mathbb{R}^n} \left| u(y, \tau) - G\left(\frac{y}{(1+\tau)^{\frac{1}{2}}}\right) (1+\tau)^{-\frac{n}{2}} \right|^2 \frac{1}{G\left(\frac{y}{(1+\tau)^{\frac{1}{2}}}\right)} dy \leq \frac{C}{(1+\tau)^{\frac{n}{2}+1}}.$$

*Proof.* We start with an energy estimate for  $w$ . Let  $\mathcal{F}(w(t)) = \int_{\mathbb{R}^n} |w - 1|^2 d\mu$ , then

$$\frac{d\mathcal{F}(w(t))}{dt} = - \int_{\mathbb{R}^n} |\nabla w|^2 G(x) dx =: -\mathcal{D}(w(t)).$$

By the Gaussian Poincaré's inequality in lemma 3.3 with measure  $d\mu = G(x)dx$ , we have

$$\int_{\mathbb{R}^n} |w - 1|^2 d\mu \leq \int_{\mathbb{R}^n} |\nabla w|^2 d\mu.$$

Therefore

$$\int_{\mathbb{R}^n} |w - 1|^2 d\mu \leq e^{-t} \int_{\mathbb{R}^n} |w_0 - 1|^2 d\mu = e^{-t} \int_{\mathbb{R}^n} |u_0 - G|^2 dx. \quad (3.15)$$

The result is obtained by changing back to the original variable,  $u(y, \tau) = v(x, t)(1 + \tau)^{-\frac{n}{2}} = w(x, t)G(x, t)(1 + \tau)^{-\frac{n}{2}}$ , namely,

$$\begin{aligned} \int_{\mathbb{R}^n} |w(t) - 1|^2 d\mu &= \int_{\mathbb{R}^n} |v(x, t) - G(x)|^2 \frac{1}{G(x)} dx \\ &= \int_{\mathbb{R}^n} \left| u(y, \tau)(1 + \tau)^{\frac{n}{2}} - G\left(\frac{y}{(1 + \tau)^{\frac{1}{2}}}\right) \right|^2 \frac{1}{G\left(\frac{y}{(1 + \tau)^{\frac{1}{2}}}\right)} \frac{dy}{(1 + \tau)^{\frac{n}{2}}} \\ &= (1 + \tau)^{\frac{n}{2}} \int_{\mathbb{R}^n} \left| u(y, \tau) - G\left(\frac{y}{(1 + \tau)^{\frac{1}{2}}}\right)(1 + \tau)^{-\frac{n}{2}} \right|^2 \frac{1}{G\left(\frac{y}{(1 + \tau)^{\frac{1}{2}}}\right)} dy \end{aligned}$$

and the right hand side with variable  $\tau$  is

$$e^{-t} \int_{\mathbb{R}^n} |u_0 - G|^2 dx = C \frac{1}{1 + \tau},$$

which completes the proof.  $\square$

*Remark 3.20.* From (3.15), instead of changing back to the variable immediately, we can use the estimate from weighted heat equation

$$\|w - 1\|_{\infty} \leq C \|w - 1\|_{L^2}.$$

We can get

$$\|w - 1\|_{\infty} \leq K e^{-t}.$$

Therefore, we get a point-wise estimate,

$$\left(1 - \frac{C}{1 + \tau}\right)(1 + \tau)^{\frac{n}{2}} G\left(\frac{y}{(1 + \tau)^{\frac{1}{2}}}\right) \leq u(y, \tau) \leq \left(1 + \frac{C}{1 + \tau}\right)(1 + \tau)^{\frac{n}{2}} G\left(\frac{y}{(1 + \tau)^{\frac{1}{2}}}\right).$$

*Remark 3.21.* One can also proceed to do the entropy estimate. We start with an entropy estimate for  $v$ . Let

$$\mathcal{E}(v(t)) = \int_{\mathbb{R}^n} \frac{v}{G} \log \frac{v}{G} d\mu = \int_{\mathbb{R}^n} v \log v dx + \frac{1}{2} \int_{\mathbb{R}^n} x^2 v + C,$$

then

$$\frac{d\mathcal{E}(v(t))}{dt} = - \int_{\mathbb{R}^n} v \left| \frac{\nabla v}{v} + x \right|^2 dx = - \int_{\mathbb{R}^n} v \left| \nabla \log\left(\frac{v}{G}\right) \right|^2 dx =: -\mathcal{I}(v(t)).$$

By using the logarithmic Sobolev inequality, i.e.  $\int f^2 \log f dx \leq \int |\nabla f|^2 dx$ , we have

$$\mathcal{E} \leq \frac{1}{2} \mathcal{I}.$$

Therefore,

$$\frac{d\mathcal{E}}{dt} + 2\mathcal{E} \leq 0 \implies \mathcal{E}(t) \leq e^{-2t}\mathcal{E}(0).$$

### 3.7. Problems.

- (1) Find the formal solution of the following problem by Fourier transform

$$\begin{aligned} i\partial_t u + \Delta u &= 0, & (x, t) \in \mathbb{R}^n \times (0, +\infty), \\ u|_{t=0} &= g(x), & x \in \mathbb{R}^n. \end{aligned}$$

- (2) Find the formal solution of the following problems

(a)

$$\begin{aligned} u_t - \Delta u + 2u &= f(x, t), & (x, t) \in \mathbb{R}^n \times (0, \infty), \\ u|_{t=0} &= \varphi(x), & x \in \mathbb{R}^n. \end{aligned}$$

(b)

$$\begin{aligned} u_t - u_{xx} + xu &= 0, & (x, t) \in \mathbb{R} \times (0, \infty), \\ u|_{t=0} &= \varphi(x), & x \in \mathbb{R}. \end{aligned}$$

(c)

$$\begin{aligned} u_t &= a^2 u_{xx}, & (x, t) \in (0, +\infty) \times (0, \infty), \\ u|_{t=0} &= 0, & x \in (0, +\infty), \\ u_x|_{x=0} &= -1, & t > 0. \end{aligned}$$

- (3) Find the Green's function of half line problem

$$\begin{aligned} u_t - u_{xx} &= f, & x \in (0, +\infty), t > 0, \\ u|_{t=0} &= \varphi, & x \in (0, +\infty), \\ u_x|_{x=0} &= 0, & t > 0. \end{aligned}$$

And give the formal solution formula of this problem.

- (4)  $\Omega$  is a bounded open subset of  $\mathbb{R}^n$ ,  $Q = \Omega \times (0, T]$ . If  $u \in C^{2,1}(Q) \cap C(\bar{Q})$  is a solution of the following initial boundary value problem

$$\begin{aligned} u_t - \Delta u &= f(x, t), & (x, t) \in Q, \\ u|_{t=0} &= \varphi(x), & x \in \Omega, \\ u|_{\partial\Omega} &= 1. \end{aligned}$$

Try to prove there exists a constant  $C$  (depending on  $T$  and  $|\Omega| = \int_{\Omega} dx$ ) such that the following inequality holds

$$\sup_{0 \leq t \leq T} \int_{\Omega} u^2(x, t) dx + \int_0^T \int_{\Omega} |\nabla u|^2 dx dt \leq C \left( \int_{\Omega} \varphi^2 dx + \int_0^T \int_{\Omega} f^2 dx dt + 1 \right).$$

(5) Find the formal solution of the following problem by using separation of variables

$$\begin{aligned} u_t &= a^2 u_{xx}, & (x, t) &\in (0, 1) \times (0, \infty), \\ u|_{t=0} &= x^2(1-x), & x &\in (0, 1), \\ u_x|_{x=0} &= u_x|_{x=1} = 0, & t &> 0. \end{aligned}$$

(6)  $\Omega$  is a bounded open subset of  $\mathbb{R}^n$ ,  $Q_T = \Omega \times [0, T)$ .  $c(x, t) \geq -c_0$  with some constant  $c_0 > 0$ . If  $u \in C(\bar{Q}_T) \cap C^{2,1}(Q_T)$  satisfies

$$\begin{aligned} u_t - a^2 \Delta u + c(x, t)u &\leq 0, & (x, t) &\in Q_T, \\ u|_{\partial_p Q_T} &\leq 0. \end{aligned}$$

Try to prove that  $u \leq 0$  in  $Q_T$ . (Hint: try to use auxiliary function  $e^{-c_0 t} u$ )

## 4. POISSON'S EQUATION

4.1. **Fundamental solution.** The Poisson's equation in  $\mathbb{R}^n$  reads

$$-\Delta u = f, \quad \text{in } \mathbb{R}^n. \quad (4.1)$$

We will first try to find some special solution formally. Since the Laplace operator is radially symmetric, it is natural to expect radially symmetric solutions. Assume  $u(x) = v(|x|) = v(r)$ , where  $r = |x|$ , then

$$u_{x_i} = v_r \frac{\partial r}{\partial x_i} = v_r \frac{x_i}{r}, \quad u_{x_i x_j} = v_{rr} \frac{x_i^2}{r^2} + v_r \left( \frac{1}{r} - \frac{x_i^2}{r^3} \right),$$

thus we can put this radially symmetric function into the homogeneous Poisson's equation  $\Delta u = 0$ , i.e.

$$\Delta u = v_{rr} + \frac{n-1}{r} v_r = 0, \quad \Rightarrow \quad (\log v_r)_r = \frac{1-n}{r}, \quad \text{in the case of } v_r \neq 0.$$

Consequently, there exist constants  $C$  and  $C'$  such that  $v_r = Cr^{1-n}$  and

$$v(r) = \begin{cases} C \log r + C', & n = 2, \\ \frac{C}{r^{n-2}} + C', & n \geq 3. \end{cases}$$

Therefore,  $u(x) = v(|x|)$  are solutions of Poisson's equation at any point but not at  $x = 0$ . Motivated by this formal computation, we give the following definition.

**Definition 12.** Let

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x|, & n = 2, \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}}, & n \geq 3, \end{cases}$$

where  $\alpha(n)$  is the volume of  $n$  dimension ball.  $\Phi(x)$  is called the **fundamental solution** of Poisson's equation.

**Properties**

- (1)  $|\nabla \Phi| \leq \frac{C}{|x|^{n-1}}$ ,  $|D^2 \Phi| \leq \frac{C}{|x|^n}$  for  $x \neq 0$ .
- (2)  $\Delta \Phi = 0$  for  $x \neq 0$  and  $\Delta \Phi(x-y) = 0$  for  $x \neq y$ ,  $\forall y \in \mathbb{R}^n$ .

*Remark 4.1.* The special choice of constants in the fundamental solution will be observed clearly in the following theorem.

Then we are able to represent the solution of Poisson's equation by using fundamental solution. More precisely we have the following theorem.

**Theorem 4.1.** *If  $f \in C_0^2(\mathbb{R}^n)$ , then  $u = \Phi * f$  is a solution of problem (4.1)*

*Proof.* First we prove that  $u \in C^2(\mathbb{R}^n)$ . In fact,

$$\frac{u(x + he_i) - u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y) \frac{f(x + he_i - y) - f(x - y)}{h} dy.$$

Due to the fact that  $f$  has compact support,  $\frac{\partial f(x-y)}{\partial x_i} = \lim_{h \rightarrow 0} \frac{f(x+he_i-y) - f(x-y)}{h}$  and  $\Phi$  is locally integrable, by letting  $h \rightarrow 0$ , we have

$$\frac{\partial u}{\partial x_i} = \int_{\mathbb{R}^n} \Phi(y) \frac{\partial f}{\partial x_i}(x-y) dy.$$

By similar discussions, we can show that  $u$  is twice differentiable and

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \int_{\mathbb{R}^n} \Phi(y) \frac{\partial^2 f}{\partial x_i \partial x_j}(x-y) dy.$$

Next we will prove  $-\Delta u = f$ .  $\forall \varepsilon > 0$  small enough,

$$\begin{aligned} -\Delta u(x) &= \int_{\mathbb{R}^n} \Phi(y) \Delta_x f(x-y) dy \\ &= \int_{B_\varepsilon(0)} \Phi(y) \Delta_x f(x-y) dy + \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Phi(y) \Delta_x f(x-y) dy \\ &=: I_\varepsilon + J_\varepsilon. \end{aligned}$$

where

$$|I_\varepsilon| \leq C \|D^2 f\|_{L^\infty} \int_{B_\varepsilon(0)} |\Phi(y)| dy \leq \begin{cases} C\varepsilon^2 |\log \varepsilon| & n = 2 \\ C\varepsilon^2 & n \geq 3 \end{cases}.$$

Integrating  $J_\varepsilon$  by parts, we have

$$J_\varepsilon = - \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \nabla_y \Phi(y) \nabla_y f(x-y) dy - \int_{\partial B_\varepsilon(0)} \Phi(y) \nabla_y f(x-y) \cdot \gamma dS_y := K_\varepsilon + L_\varepsilon,$$

where  $L_\varepsilon$  can be estimated by

$$|L_\varepsilon| \leq \|Df\|_{L^\infty} \int_{\partial B_\varepsilon(0)} |\Phi(y)| dS_y \leq \begin{cases} C\varepsilon |\log \varepsilon|, & n = 2, \\ C\varepsilon, & n \geq 3 \end{cases}.$$

In the end, we can see that  $K_\varepsilon$  contributes the main part of the calculation. When  $\varepsilon$  goes to 0, this term plays a role as a Delta function applied on  $f$ . Due to the fact that  $\Delta \Phi(y) = 0$  for  $y \neq 0$ , we have

$$K_\varepsilon = \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Delta \Phi(y) f(x-y) dy + \int_{\partial B_\varepsilon(0)} \nabla \Phi \cdot \gamma f(x-y) dS_y = \int_{\partial B_\varepsilon(0)} \nabla \Phi \cdot \gamma f(x-y) dS_y.$$

Now we can calculate that on  $\partial B_\varepsilon(0)$ ,

$$\nabla_y \Phi(y) \cdot \gamma = -\frac{1}{n\alpha(n)} \frac{y}{|y|^n} \frac{y}{|y|} = -\frac{1}{n\alpha(n)\varepsilon^{n-1}}.$$

Thus we have

$$K_\varepsilon = -\frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B_\varepsilon(0)} f(x-y) dS_y = -\frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B_\varepsilon(x)} f(y) dS_y.$$

In the end, taking  $\varepsilon \rightarrow 0$  gives

$$K_\varepsilon \rightarrow f(x).$$

□

*Remark 4.2.* From the above proof, the delicate choosing of the constants that appear in the definition of the fundamental solution is now apparent and understandable.

By using the same method, we can prove that  $-\Delta\Phi = \delta_0(x)$  in the sense of distribution.

**Theorem 4.2.**

$$\Phi(x, y) = \Phi(x - y) = \begin{cases} -\frac{1}{2\pi} \log |x - y|, & n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x - y|^{n-2}}, & n \geq 3 \end{cases} \quad (4.2)$$

is a solution of

$$-\Delta\Phi = \delta_y(x)$$

in the sense of distribution. More precisely,  $\forall \varphi \in C_0^\infty(\mathbb{R}^n)$ , it holds

$$\langle -\Delta\Phi(x - y), \varphi(x) \rangle = - \int_{\mathbb{R}^n} \Phi(x - y) \Delta\varphi(x) dx = \varphi(y) = \langle \delta_y(x), \varphi(x) \rangle = \varphi(y).$$

**4.2. Properties of harmonic functions.** Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ .

**Definition 13.** If  $\Delta u = 0$  in  $\Omega$  with  $u \in C^2(\Omega)$ , then  $u$  is called a **harmonic function** in  $\Omega$ .

4.2.1. *Mean value theorem.*

**Theorem 4.3.** If  $u \in C^2(\Omega)$  is harmonic, then  $\forall$  ball  $B(x, r) \in \Omega$ , it holds that

$$u(x) = \int_{\partial B(x, r)} u dS_y = \int_{B(x, r)} u dy. \quad (4.3)$$

*Proof.* Let

$$w(r) = \int_{\partial B(x, r)} u(y) dS_y = \int_{\partial B(0, 1)} u(x + rz) dS_z$$

Then by taking the derivative with respect to  $r$ , we have

$$\begin{aligned} w'(r) &= \int_{\partial B(0, 1)} \nabla u(x + rz) \cdot z dS_z \\ &= \int_{\partial B(x, r)} \nabla u(y) \cdot \frac{y - x}{r} dS_y = \frac{r}{n|B(x, r)|} \int_{B(x, r)} \Delta u(y) dy = 0, \end{aligned}$$

which implies that  $w(r)$  is a constant. Thus we have

$$w(r) = \lim_{s \rightarrow 0} w(s) = \lim_{s \rightarrow 0} \int_{\partial B(x, s)} u(y) dS_y = u(x).$$

For the mean value on  $B(x, r)$ , we know that

$$\begin{aligned} \int_{B(x, r)} u(y) dy &= \int_0^r \left( \int_{\partial B(x, s)} u(y) dS_y \right) ds \\ &= u(x) \int_0^r n\alpha(n) s^{n-1} ds = \alpha(n) r^n u(x), \end{aligned}$$

which is exactly

$$u(x) = \int_{B(x,r)} u(y) dy.$$

□

**Theorem 4.4.** *If  $u \in C^2(\Omega)$  such that*

$$u(x) = \int_{\partial B(x,r)} u(y) dS_y, \quad \forall B(x,r) \subset \Omega,$$

*then  $u$  is harmonic in  $\Omega$ , i.e.,  $\Delta u = 0$  in  $\Omega$ .*

*Proof.* If  $\Delta u \not\equiv 0$ , there must exist a ball  $B(x,r) \subset \Omega$  such that  $\Delta u > 0$  in  $B(x,r)$ . On the other hand,

$$0 = w'(r) = \frac{r}{n} \int_{B(x,r)} \Delta u(y) dy > 0,$$

which gives a contradiction. □

#### 4.2.2. Strong maximum principle.

**Theorem 4.5.** *If  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  is harmonic in  $\Omega$ , then*

- (1)  $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$
- (2) *If  $\Omega$  is connected and  $\exists x_0 \in \Omega$  such that*

$$u(x_0) = \max_{\bar{\Omega}} u(x),$$

*then  $u$  is constant within  $\Omega$ .*

*Proof.* The first statement is easy, we only prove that second one here.

Let

$$U_M = \{x \in \bar{\Omega} | u(x) = M = \max_{\bar{\Omega}} u(x)\} \subset \bar{\Omega}.$$

Then the set  $U_M$  is nonempty, furthermore it is a closed set because of the continuity of  $u$ . If  $U_M \subset\subset \bar{\Omega}$ , then  $\exists x_1 \in \partial U_M \subset \Omega$  and  $\forall 0 < r < \text{dist}(x_1, \partial\Omega)$  such that  $u(x_1) = M$  and

$$M = u(x_1) = \int_{B(x_1,r)} u(y) dy < M,$$

where the mean value property has been applied. We have arrived at a contradiction. □

**Corollary 4.1.** *If  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  is harmonic and  $u \geq 0$  on  $\partial\Omega$ , then  $u \geq 0$  in  $\Omega$ .*

**Corollary 4.2.** *(Uniqueness) Dirichlet boundary value problem  $-\Delta u = f$  in  $\Omega$  and  $u = g$  on  $\partial\Omega$  admits at most one  $C^2(\Omega) \cap C(\bar{\Omega})$ - solution.*

### 4.2.3. Regularity.

**Theorem 4.6.** *If  $u \in C(\Omega)$  satisfies mean value property for all ball  $B(x, r)$  in  $\Omega$ , then  $u \in C^\infty(\Omega)$ .*

*Remark 4.3.* The smoothness up to  $\partial\Omega$  is usually not true, which depends on the regularity of the boundary.

*Proof.* We use mollification to the proof of regularity. Let  $j(x) \in C_0^\infty(\mathbb{R}^n)$  and  $\int_{\mathbb{R}^n} j(x) dx = 1$ .  $\forall \varepsilon > 0$ , we introduce the scaling

$$j_\varepsilon(x) = \frac{1}{\varepsilon^n} j\left(\frac{x}{\varepsilon}\right).$$

Let

$$\Omega_\varepsilon = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \varepsilon\}.$$

Therefore  $\forall x \in \Omega_\varepsilon$ ,  $u_\varepsilon(x) = j_\varepsilon(x) * u(x) \in C^\infty(\Omega_\varepsilon)$ , and with the help of the mean value property, we have

$$\begin{aligned} u_\varepsilon(x) &= \int_{B(x, \varepsilon)} \frac{1}{\varepsilon^n} j\left(\frac{x-y}{\varepsilon}\right) u(y) dy \\ &= \frac{1}{\varepsilon^n} \int_0^\varepsilon \left[ j\left(\frac{r}{\varepsilon}\right) \int_{\partial B(x, r)} u(y) dS_y \right] dr \\ &= \frac{1}{\varepsilon^n} u(x) \int_0^\varepsilon j\left(\frac{r}{\varepsilon}\right) n\alpha(n) r^{n-1} dr \\ &= u(x) \int_{B(0, \varepsilon)} j_\varepsilon(y) dy = u(x). \end{aligned}$$

Thus  $u(x) = u_\varepsilon(x) \in C^\infty(\Omega_\varepsilon)$ ,  $\forall \varepsilon > 0$ . □

### 4.2.4. Liouville theorem.

**Theorem 4.7.** *If  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  is harmonic and bounded, then  $u$  is a constant.*

*Proof.* \*\*\* The proof will use local regularity estimates for harmonic function which was not talked about in this course.  $\forall x_0 \in \mathbb{R}^n$ ,  $r > 0$ ,

$$|Du(x_0)| \leq \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x_0, r))} \leq \frac{C_1 \alpha(n)}{r} \|u\|_{L^\infty(\mathbb{R}^n)} \rightarrow 0, \text{ as } r \rightarrow \infty.$$

Then  $Du \equiv 0$ , which implies  $u$  is a constant. □

**Corollary 4.3.** *If  $f \in C_0^2(\mathbb{R}^n)$ ,  $n \geq 3$ , then any bounded solution of  $-\Delta u = f$  in  $\mathbb{R}^n$  has the form*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy + C.$$

*Proof.* First we know that  $\int_{\mathbb{R}^n} \Phi(x-y) f(y) dy$  is a bounded solution since  $\Phi(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . If there is another bounded solution  $\tilde{u}$ , then  $w = u - \tilde{u}$  is harmonic, thus by Liouville's theorem,  $w$  is a constant. □

**4.3. Green's Function.** The main goal of this subsection is to get the representation formula for the solution of boundary value problem

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= \varphi. \end{aligned} \tag{4.4}$$

Is the fundamental solution useful in getting a solution formula to the above Dirichlet problem?

Let's start from a formal calculation,  $\forall x \in \Omega$ ,

$$\begin{aligned} u(x) &= \langle \delta_x(y), u(y) \rangle = \langle -\Delta_y \Phi(x, y), u(y) \rangle \\ &\sim - \int_{\Omega} \Delta_y \Phi(x, y) u(y) dy \\ &= \int_{\Omega} \Phi(x, y) (-\Delta_y u(y)) dy - \int_{\partial\Omega} \nabla_y \Phi(x, y) \cdot \gamma u(y) dS_y + \int_{\partial\Omega} \Phi(x, y) \nabla_y u(y) \cdot \gamma dS_y. \end{aligned}$$

Then formally, if  $u|_{\partial\Omega} = \varphi$  and  $-\Delta u = f$ , we have

$$u(x) = \int_{\Omega} \Phi(x, y) f(y) dy - \int_{\partial\Omega} \nabla_y \Phi(x, y) \cdot \gamma \varphi(y) dS_y + \int_{\partial\Omega} \Phi(x, y) \nabla_y u(y) \cdot \gamma dS_y,$$

where however the last term is still unknown. We will try to consider another function  $G(x, y)$  to replace the fundamental solution  $\Phi(x, y)$ . And this  $G(x, y)$  satisfies, for any fixed  $y \in \Omega$ ,

$$\begin{aligned} -\Delta_y G(x, y) &= \delta_x(y), \quad \forall y \in \Omega, \\ G(x, y)|_{y \in \partial\Omega} &= 0. \end{aligned}$$

A good candidate of  $G(x, y)$  is  $\Phi(x, y) + g(x, y)$  with  $g(x, y)$  satisfying the following Dirichlet boundary value problem

$$\begin{aligned} -\Delta_y g(x, y) &= 0, \quad \forall x, y \in \Omega, \\ g|_{y \in \partial\Omega} &= -\Phi(x, y)|_{y \in \partial\Omega}. \end{aligned}$$

Once we can solve the above problem for  $g(x, y)$ , we will have the solution representation of (4.4),

$$u(x) = \int_{\Omega} G(x, y) f(y) dy - \int_{\partial\Omega} \nabla_y G(x, y) \cdot \gamma \varphi(y) dS_y.$$

We will give a proof of the above discussion after the following definition.

**Definition 14.** (Green's function)

$$G(x, y) = \Phi(x, y) + g(x, y)$$

is called the **Green's function** of (4.4), where  $g(x, y) \in C^2(\Omega \times \Omega)$  is a solution of the following boundary value problem

$$\begin{aligned} -\Delta_y g(x, y) &= 0, \quad \forall x, y \in \Omega, \\ g(x, y)|_{y \in \partial\Omega} &= -\Phi(x, y)|_{y \in \partial\Omega}. \end{aligned}$$

**Theorem 4.8.** *If  $\Omega$  is an open subset of  $\mathbb{R}^n$ ,  $\partial\Omega$  is piecewise smooth and  $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ , then  $\forall x \in \Omega$ ,*

$$u(x) = \int_{\Omega} \Phi(x, y)(-\Delta_y u(y)) dy - \int_{\partial\Omega} \nabla_y \Phi(x, y) \cdot \gamma u(y) dS_y + \int_{\partial\Omega} \Phi(x, y) \nabla_y u(y) \cdot \gamma dS_y \quad (4.5)$$

*Proof.*  $\forall \varepsilon > 0$  small enough, we have

$$\begin{aligned} & \int_{\Omega} \Phi(x, y)(-\Delta_y u(y)) dy = \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega \setminus B_\varepsilon(x)} \Phi(x, y)(-\Delta_y u(y)) dy \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega \setminus B_\varepsilon(x)} -\Delta_y \Phi(x, y) u(y) dy - \lim_{\varepsilon \rightarrow 0^+} \int_{\partial\Omega} (\Phi(x, y) \nabla u(y) \cdot \gamma - \nabla \Phi(x, y) \cdot \gamma u(y)) dS_y \\ & \quad - \lim_{\varepsilon \rightarrow 0^+} \int_{\partial B(x, \varepsilon)} (\Phi(x, y) \nabla u(y) \cdot \gamma - \nabla \Phi(x, y) \cdot \gamma u(y)) dS_y \\ &= 0 - \lim_{\varepsilon \rightarrow 0^+} \int_{\partial\Omega} (\Phi(x, y) \nabla u(y) \cdot \gamma - \nabla \Phi(x, y) \cdot \gamma u(y)) dS_y + u(x), \end{aligned}$$

where we have used the facts that

$$\left| \int_{\partial B(x, \varepsilon)} \Phi(x, y) \nabla u(y) \cdot \gamma dS_y \right| \leq C\varepsilon \max_{\partial B(x, \varepsilon)} |\nabla u| \rightarrow 0, \text{ as } \varepsilon \rightarrow 0,$$

and

$$\int_{\partial B(x, \varepsilon)} u(y) \nabla \Phi(x, y) \cdot \gamma dS_y = \int_{\partial B(x, \varepsilon)} u(y) dS_y \rightarrow u(x), \text{ as } \varepsilon \rightarrow 0.$$

□

**Theorem 4.9.** *(Green's function is symmetric in its two variables)*

$$G(x, y) = G(y, x), \quad \forall x, y \in \Omega.$$

\*\*\*Sketch of the proof. The technical point is the same as the proof of theorem 4.8.  $\forall \varepsilon > 0$  small enough such that  $B(x, \varepsilon) \cup B(y, \varepsilon) \subset \Omega$  and  $B(x, \varepsilon) \cap B(y, \varepsilon) = \emptyset$ , take  $\Omega_\varepsilon = \Omega \setminus (B(x, \varepsilon) \cup B(y, \varepsilon))$ . Notice that  $G(x, z) = G(y, z) = 0$  on  $z \in \partial\Omega$ ,

$$\begin{aligned} 0 &= \int_{\Omega_\varepsilon} (G(y, z) \Delta_z G(x, z) - G(x, z) \Delta_z G(y, z)) dz \\ &= \int_{\partial\Omega_\varepsilon} (G(y, z) \nabla_z G(x, z) \cdot \gamma - G(x, z) \nabla_z G(y, z) \cdot \gamma) dS_z \\ &= \int_{\partial B(x, \varepsilon) \cup \partial B(y, \varepsilon)} (G(y, z) \nabla_z G(x, z) \cdot \gamma - G(x, z) \nabla_z G(y, z) \cdot \gamma) dS_z. \end{aligned}$$

We take  $\partial B(y, \varepsilon)$  as an example,

$$\begin{aligned} & \left| \int_{\partial B(y, \varepsilon)} G(y, z) \nabla_z G(x, z) \cdot \gamma dS_z \right| \leq C(\varepsilon + \varepsilon^{n-1}) \rightarrow 0, \\ & - \int_{\partial B(y, \varepsilon)} G(x, z) \nabla_z G(y, z) \cdot \gamma dS_z = \int_{\partial B(y, \varepsilon)} G(x, z) dS_z + o(\varepsilon^{n-1}) \rightarrow -G(x, y). \end{aligned}$$

We can proceed the discussion in a similar manner for the integral on  $\partial B(x, \varepsilon)$  and conclude that it converges to  $G(y, x)$ .

4.3.1. *Half space problem.* The half space we study here is  $\mathbb{R}_+^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n | x_n > 0\}$ .

$\forall x = (x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}_+^n$ , we call  $\tilde{x} = (x_1, \dots, x_{n-1}, -x_n)$  is  $x$ 's reflection in the plane  $\{x_n = 0\}$ .

We study the following boundary value problem

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \mathbb{R}_+^n, \\ u|_{\partial\mathbb{R}_+^n} &= \varphi. \end{aligned}$$

Our goal here is to find Green's function  $G(x, y)$  of this problem and write the solution by using formula

$$u(x) = \int_{\Omega} G(x, y)f(y)dy - \int_{\partial\Omega} \nabla_y G(x, y) \cdot \gamma\varphi(y)dS_y.$$

$\forall x \in \mathbb{R}_+^n$ , the Green's function should be a solution of

$$\begin{aligned} -\Delta_y G &= \delta_x(y), \quad y \in \mathbb{R}_+^n, \\ G|_{y \in \partial\mathbb{R}_+^n} &= 0. \end{aligned}$$

Then the Green's function of half space problem is easy to obtain, i.e.,

$$G(x, y) = \Phi(x, y) - \Phi(\tilde{x}, y), \quad x, y \in \mathbb{R}_+^n, x \neq y,$$

and

$$\frac{\partial G}{\partial y_n}(x, y) = \frac{\partial \Phi}{\partial y_n}(y - x) - \frac{\partial \Phi}{\partial y_n}(y - \tilde{x}) = \frac{-1}{n\alpha(n)} \left( \frac{y_n - x_n}{|y - x|^n} - \frac{y_n + x_n}{|y - \tilde{x}|^n} \right).$$

Therefore,  $\forall y \in \partial\mathbb{R}_+^n$ ,

$$\frac{\partial G}{\partial \gamma}(x, y) = -\frac{\partial G}{\partial y_n}(x, y) = -\frac{2x_n}{n\alpha(n)} \frac{1}{|x - y|^n}.$$

Then the solution of the boundary value problem in case  $f \equiv 0$  can be represented by

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}_+^n} \frac{\varphi(y)}{|x - y|^n} dy, \quad \forall x \in \mathbb{R}_+^n,$$

which is called the **Poisson's formula** of half space problem.

The function

$$K(x, y) := \frac{2x_n}{n\alpha(n)} \frac{1}{|x - y|^n}, \quad x \in \mathbb{R}_+^n, y \in \partial\mathbb{R}_+^n$$

is called the **Poisson kernel** for  $\mathbb{R}_+^n$ .

**Theorem 4.10.** Assume  $\varphi \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$  and  $u$  is defined by the Poisson's formula. Then  $u \in C^\infty(\mathbb{R}_+^n) \cap L^\infty(\mathbb{R}_+^n)$ ,  $-\Delta u = 0$  in  $\mathbb{R}_+^n$  and  $\forall x^0 \in \partial\mathbb{R}_+^n$ ,

$$\lim_{x \in \mathbb{R}_+^n, x \rightarrow x^0} u(x) = \varphi(x^0).$$

*Proof.* The facts that  $-\Delta u = 0$  and  $\forall x \in \mathbb{R}_+^n$ ,

$$\int_{\partial\mathbb{R}_+^n} K(x, y) dy = 1,$$

are easy to check.

Since  $\forall x \neq y$ ,  $K(x, y)$  is a smooth function in  $x$ , we know directly that  $u \in C^\infty(\mathbb{R}_+^n)$  and

$$\Delta u(x) = \int_{\partial\mathbb{R}_+^n} \Delta_x K(x, y) \varphi(y) dy = 0, \quad \forall x \in \mathbb{R}_+^n.$$

For boundary condition,  $\forall x^0 \in \partial\mathbb{R}_+^n$ ,  $\forall \varepsilon > 0$ , choose  $\delta > 0$  small enough such that  $\forall y \in \partial\mathbb{R}_+^n$  and  $|y - x^0| < \delta$ , we have

$$|\varphi(y) - \varphi(x^0)| < \varepsilon.$$

Then  $\forall x \in \mathbb{R}_+^n$  and  $|x - x^0| < \delta/2$ , we have

$$\begin{aligned} |u(x) - \varphi(x^0)| &= \left| \int_{\partial\mathbb{R}_+^n} K(x, y) (\varphi(y) - \varphi(x^0)) dy \right| \\ &\leq \int_{\partial\mathbb{R}_+^n \cap B(x^0, \delta)} K(x, y) |\varphi(y) - \varphi(x^0)| dy + \int_{\partial\mathbb{R}_+^n \setminus B(x^0, \delta)} K(x, y) |\varphi(y) - \varphi(x^0)| dy \\ &\leq \varepsilon + 2\|\varphi\|_{L^\infty} \int_{\partial\mathbb{R}_+^n \setminus B(x^0, \delta)} K(x, y) dy \\ &\leq \varepsilon + \frac{2^{n+1}\|\varphi\|_{L^\infty} x_n}{n\alpha(n)} \int_{\partial\mathbb{R}_+^n \setminus B(x^0, \delta)} \frac{1}{|y - x^0|^n} dy \rightarrow 0, \quad \text{as } x_n \rightarrow 0^+. \end{aligned}$$

□

4.3.2. *Problem in a ball.* We will give an exact formula for the Green's function in a ball.

$\forall x \in B^n(0, 1)$ , the  $n$  dimensional unit ball. We need that  $G(x, y) = 0$ ,  $\forall y \in \partial B^n(0, 1)$ . Let  $\tilde{x}$  be the inversion of  $x$ , i.e.,  $\tilde{x} = \frac{x}{|x|^2}$ , thus

$$|\tilde{x} - y| \cdot |x| = |x - y|, \quad \forall y \in \partial B^n(0, 1)$$

and in order that  $G(x, y)$  satisfies the zero boundary condition, we set

$$G(x, y) = \Phi(|x - y|) - \Phi(|y - x|) = \Phi(|y - x|) - \Phi(|x| \cdot |y - \tilde{x}|), \quad \forall y \in \partial B^n(0, 1).$$

Since  $\Phi$  is the fundamental solution,

$$-\Delta_y \Phi(|x| \cdot |y - \tilde{x}|) = 0, \quad \forall y \neq \tilde{x}.$$

As a consequence,

$$G(x, y) = \Phi(|y - x|) - \Phi(|x| \cdot |y - \tilde{x}|), \quad \forall y \in B^n(0, 1)$$

is called the Green's function for  $B^n(0, 1)$ .

Now we will give the Poisson's formula for  $B^n(0, 1)$ .

$$\begin{aligned} -\Delta u &= 0, \quad \text{in } B^n(0, 1), \\ u|_{\partial B(0,1)} &= h. \end{aligned}$$

By Green's formula the solution is

$$u(x) = - \int_{\partial B(0,1)} h(y) \nabla G(x, y) \cdot \gamma dS_y.$$

We will explicitly calculate this formula.

$$\begin{aligned} \nabla_y \Phi(y - x) &= -\frac{1}{n\alpha(n)} \frac{y - x}{|x - y|^n}, \\ \nabla_y \Phi(|x|(y - \frac{x}{|x|^2})) &= -\frac{1}{n(n-2)\alpha(n)} \nabla_y \frac{1}{|x|^{n-2} |y - \frac{x}{|x|^2}|^{n-2}} \\ &= \frac{-1}{n\alpha(n)} \frac{1}{|x|^{n-2}} \frac{y - \frac{x}{|x|^2}}{|y - \frac{x}{|x|^2}|^n} = \frac{-1}{n\alpha(n)} \frac{y|x|^2 - x}{[|x|(y - \frac{x}{|x|^2})]^n} \\ &= \frac{-1}{n\alpha(n)} \frac{y|x|^2 - x}{|x - y|^n}, \end{aligned}$$

where we have used the fact that  $y \in \partial B(0, 1)$  and  $|x| \cdot |y - \frac{x}{|x|^2}| = |x - y|$ . Furthermore,

$$\begin{aligned} \nabla_y G(x, y) \cdot \gamma|_{\partial B(0,1)} &= \frac{-1}{n\alpha(n)} \left( \frac{y - x}{|x - y|^n} - \frac{y|x|^2 - x}{|x - y|^n} \right) \cdot y \Big|_{y \in \partial B(0,1)} \\ &= \frac{-1}{n\alpha(n)} \frac{|y|^2 - x \cdot y - |y|^2|x|^2 + x \cdot y}{|x - y|^n} \Big|_{|y|=1} \\ &= \frac{-1}{n\alpha(n)} \frac{|y|^2(1 - |x|^2)}{|x - y|^n} \Big|_{|y|=1} = \frac{-1}{n\alpha(n)} \frac{1 - |x|^2}{|x - y|^n}. \end{aligned}$$

Thus the solution formula is

$$u(x) = \frac{1 - |x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{h(y)}{|x - y|^n} dS_y.$$

For problems on  $B(0, r)$ , by doing scaling  $\tilde{u}(x) = u(rx)$ ,  $\tilde{h}(x) = h(rx)$ , we will have the **Poisson's formula**

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{h(y)}{|x - y|^n} dS_y, \quad \forall x \in B(0, r). \quad (4.6)$$

We call

$$\frac{r^2 - |x|^2}{n\alpha(n)r} \frac{1}{|x - y|^n}$$

the **Poisson kernel** for  $B(0, r)$ .

**Theorem 4.11.** *If  $h \in C(\partial B(0, r))$ , then the function  $u$  given by (4.6) is in  $C^\infty(B(0, r))$ ,  $-\Delta u = 0$  and  $\lim_{x \rightarrow x^0} u(x) = h(x^0)$ ,  $\forall x^0 \in \partial B$ .*

**4.4. Maximum principle.** Here we consider more general equations. Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^n$  and

$$Lu = -\Delta u + c(x)u, \quad \text{in } \Omega.$$

**Theorem 4.12.** (*Weak maximum principle*) Let  $0 \leq c(x) \leq \bar{c}$  in  $\Omega$ , if  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  and  $Lu \leq 0$  in  $\Omega$ , then

$$\max_{\bar{\Omega}} u(x) \leq \max_{\partial\Omega} u^+(x),$$

where  $u^+(x) = \max\{u(x), 0\}$ .

*Proof.* Assume  $Lu < 0$  in  $\Omega$ . If  $\exists x_0 \in \Omega$  such that

$$0 \leq u(x_0) = \max_{\Omega} u,$$

then

$$-\Delta u|_{x=x_0} + c(x_0)u(x_0) \geq 0,$$

which is a contradiction.

If  $Lu \leq 0$  in  $\Omega$ , we introduce an auxiliary function

$$w(x) = u(x) + \varepsilon e^{ax_1},$$

where  $a$  is chosen such that  $-a^2 + \bar{c} < 0$ . Therefore

$$Lw = Lu + \varepsilon e^{ax_1}(-a^2 + c(x)) < 0.$$

By using the above discussion, we have  $\max_{\bar{\Omega}} w \leq \max_{\partial\Omega} w^+$ , then the weak maximum principle follows by taking  $\varepsilon \rightarrow 0$ .

□

*Remark 4.4.* If  $c \equiv 0$ , then  $\max_{\partial\Omega} u^+$  in the theorem can be replaced by  $\max_{\partial\Omega} u$ .

*Remark 4.5.* If  $Lu \geq 0$  in  $\Omega$ , then  $\min_{\bar{\Omega}} u \geq \min_{\partial\Omega}(-u^-)$  where  $u^-(x) = \max\{-u(x), 0\}$ .

Next, by using the above weak maximum principle, we will get a maximum estimate for the following boundary value problem

$$\begin{aligned} -\Delta u &= f, & \text{in } \Omega, \\ u &= \varphi, & \text{on } \partial\Omega. \end{aligned} \tag{4.7}$$

**Theorem 4.13.** If  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  is a solution of (4.7), then

$$\max_{\bar{\Omega}} |u| \leq \Phi + CF,$$

where  $\Phi = \max_{\partial\Omega} |\varphi|$ ,  $F = \sup_{\bar{\Omega}} |f|$ ,  $C$  is a constant depending on  $n$  and  $\text{diam}\Omega$ .

*Proof.* Without loss of generality, let  $x = 0 \in \Omega$  and

$$w(x) = \pm u + \frac{F}{2n}(d^2 - |x|^2) + \Phi,$$

then

$$-\Delta w = \pm f + F \geq 0, \quad w|_{\partial\Omega} \geq \Phi \pm \varphi \geq 0.$$

By comparison principle, we have  $w \geq 0$  in  $\bar{\Omega}$ , which implies

$$\max_{\Omega} |u| \leq \Phi + \frac{F}{2n}d^2.$$

□

**4.5. Variational problem.** We show in this part that the solvability of the boundary value problem of Poisson's equation is equivalent to the solvability of a variational problem. Namely

$$\begin{aligned} -\Delta u &= f, & \text{in } \Omega, \\ u &= g, & \text{on } \partial\Omega \end{aligned} \tag{4.8}$$

is equivalent to the following problem

$$\begin{aligned} J(u) &= \inf_{v \in M_g} J(v), \\ J(v) &= \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \int_{\Omega} f v dx, \\ M_g &= \{v \in C^1(\bar{\Omega}) | v = g \text{ on } \partial\Omega\}. \end{aligned} \tag{4.9}$$

4.5.1. *Dirichlet principle.*

**Theorem 4.14.** (*Dirichlet principle*) Assume  $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ , then  $u$  is a solution of (4.8) if and only if  $u$  is a solution of (4.9).

*Proof.* “ $\Rightarrow$ ”.  $\forall v \in M_g$ , we choose  $u - v$  as test function in (4.8),

$$\int_{\Omega} -\Delta u(u - v) = \int_{\Omega} f(u - v).$$

Integration by parts with boundary condition  $u - v = 0$  on  $\partial\Omega$  shows

$$\int_{\Omega} \nabla u \cdot \nabla(u - v) = \int_{\Omega} f(u - v).$$

Equivalently,

$$\int_{\Omega} |\nabla u|^2 - \int_{\Omega} f u = \int_{\Omega} \nabla u \cdot \nabla v - \int_{\Omega} f v \leq \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v.$$

Then we have

$$\frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} f u \leq \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v,$$

which is

$$J(u) \leq J(v), \quad \forall v \in M_g.$$

“ $\Leftarrow$ ”  $\forall v \in M_0$ , we have  $u + \varepsilon v \in M_g$ . Let  $j(\varepsilon) = J(u + \varepsilon v)$ , since  $u$  is a solution of (4.9), we know that  $j'(\varepsilon)|_{\varepsilon=0} = 0$ . More precisely,

$$\begin{aligned} & \frac{d}{d\varepsilon} \left[ \int_{\Omega} \frac{1}{2} |\nabla(u + \varepsilon v)|^2 - \int_{\Omega} f(u + \varepsilon v) \right] \Big|_{\varepsilon=0} \\ &= \int_{\Omega} \nabla(u + \varepsilon v)|_{\varepsilon=0} \cdot \nabla v - \int_{\Omega} f v = \int_{\Omega} \nabla u \cdot \nabla v - \int_{\Omega} f v = \int_{\Omega} (-\Delta u - f)v, \end{aligned}$$

which holds for any  $v \in C_0^1(\Omega)$ . Thus  $u$  is a solution of (4.9).  $\square$

The equation  $-\Delta u = f$  in  $\Omega$  is called the **Euler-Lagrange equation** of the variational problem (4.9).

In the 19th century, it is thought that variational problem always has a solution in the given set of functions. But Weierstrass gave an example which shows that the infimum can not be achieved by a function in the given set, i.e., the minimizer doesn't exist. Here is the example,

**Example 12.** \*\*\* (Weierstrass) Variational problem. Let  $M = \{\varphi(x) \in C[0, 1] | \varphi'(x) \text{ is continuous except finite removable or jump discontinuous points and } \varphi(0) = 1, \varphi(1) = 0\}$ . The functional is

$$J(\varphi) = \int_0^1 [1 + (\varphi')^2]^{\frac{1}{4}} dx.$$

It is obvious that  $\inf_{\varphi \in M} J(\varphi) = 1$ . In fact, we only need to prove  $\forall \delta > 0, \exists \varphi_{\delta} \in M$  such that

$$J(\varphi_{\delta}) \leq 1 + \delta,$$

where we can choose

$$\varphi_{\delta} = \begin{cases} \frac{1}{\delta^2}(\delta^2 - x), & 0 \leq x \leq \delta^2 \\ 0, & \delta^2 < x \leq 1 \end{cases}.$$

On the other hand, we couldn't find any  $\varphi \in M$  such that  $J(\varphi) = 1$ . Otherwise,  $\varphi' = 0$  a.e., then  $\varphi \equiv C$ , which contradicts with  $\varphi(0) = 1, \varphi(1) = 0$ .

Another fact is that even if the boundary value problem (4.8) has a solution in  $C^2(\Omega) \cap C(\bar{\Omega})$ , it may not be obtained by solving the variational problem (4.9). Here is an example by Hadamard.

**Example 13.** \*\*\*  $\Omega = B(0, 1), f \equiv 0, \varphi(\theta) = \sum_{n=1}^{\infty} \frac{\sin n^4 \theta}{n^2} \in C(\partial\Omega), 0 \leq \theta \leq 2\pi$ .

We know that (4.8) has a unique solution  $u_0 \in C(\bar{\Omega}) \cap C^2(\Omega)$  with expression

$$u_0(\rho, \theta) = \sum_{n=1}^{\infty} \frac{\sin n^4 \theta}{n^2} \rho^{n^4}.$$

On the other hand we can prove that

$$J(u_0) = +\infty.$$

In fact,

$$\begin{aligned} J(u_0) &= \lim_{r \rightarrow 1^-} \int \int_{\rho \leq r} |\nabla u_0|^2 dx dy = \lim_{r \rightarrow 1^-} \int \int_{\rho \leq r} \left[ \left( \frac{\partial u_0}{\partial \rho} \right)^2 + \frac{1}{\rho^2} \left( \frac{\partial u_0}{\partial \theta} \right)^2 \right] \rho d\rho d\theta \\ &= \lim_{r \rightarrow 1^-} 2\pi \int_0^r \sum_{n=1}^{\infty} n^4 \rho^{2n^4-1} d\rho = \lim_{r \rightarrow 1^-} \pi \sum_{n=1}^{\infty} r^{2n^4} = +\infty. \end{aligned}$$

**4.6. Sobolev space  $H^1(\Omega)$  and  $H_0^1(\Omega)$ .** The Sobolev space  $H^1(\Omega)$  is defined by

$$H^1(\Omega) = \{u \in \mathcal{D}'(\Omega) | u, Du \in L^2(\Omega)\},$$

where the inner product and the norm are given respectively by

$$\langle u, v \rangle_{H^1} = \int_{\Omega} uv + \int_{\Omega} \nabla u \cdot \nabla v, \quad \|u\|_{H^1} = \|u\|_{L^2} + \|\nabla u\|_{L^2}.$$

$H^1$  is a Hilbert space.

$H_0^1(\Omega)$  is the completion of  $C_0^\infty(\Omega)$  with  $H^1$ -norm, so it is a subspace of  $H^1(\Omega)$ .

*Remark 4.6.* The Poincaré's inequality implies that  $\|u\|_{H^1}$  and  $\|\nabla u\|_{L^2}$  are equivalent norms in  $H^1$  space.

\*\*\*\*\*The most important property for Sobolev spaces is the compact embedding theorem. For bounded  $\Omega$  with uniform cone condition,  $H^1(\Omega)$  is compactly embedded in  $L^2(\Omega)$ . Therefore,  $(-\Delta)^{-1}$  with homogenous Dirichlet boundary condition is a compact operator in  $L^2(\Omega)$ , since

$$(-\Delta)^{-1} : L^2(\Omega) \rightarrow H^1(\Omega) \hookrightarrow L^2(\Omega).$$

**Definition 15.** If  $\exists u \in H_0^1(\Omega)$  such that

$$J(u) = \inf_{v \in H_0^1} \left( \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} f v \right)$$

we call  $u$  is a solution of (4.9).

**Definition 16.** If  $\exists u \in H_0^1$  such that  $\forall v \in H_0^1(\Omega)$ ,

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f v,$$

then we call  $u$  a weak solution of (4.8).

**Theorem 4.15.** If  $u \in H_0^1(\Omega)$ , then  $u$  is a weak solution of (4.8) if and only if  $u$  is a solution of (4.9).

The proof of this theorem is left to readers.

**4.7. Solvability of variational problem.** Our goal in this subsection is to prove the unique solvability of variational problem (4.9).

**Theorem 4.16.** *Solution of (4.9) in  $H_0^1(\Omega)$  is unique.*

*Proof.* Let  $u_1, u_2 \in H_0^1(\Omega)$  be two solutions of (4.9), i.e.,

$$J(u_1) = J(u_2) = m = \inf_{v \in H_0^1(\Omega)} J(v),$$

then

$$0 = \frac{1}{2} \int_{\Omega} |\nabla u_1|^2 - \frac{1}{2} \int_{\Omega} |\nabla u_2|^2 - \int_{\Omega} (u_1 - u_2)f.$$

Noticing the fact that

$$\left| \frac{\nabla(u_1 - u_2)}{2} \right|^2 + \left| \frac{\nabla(u_1 + u_2)}{2} \right|^2 = \frac{1}{2} |\nabla u_1|^2 + \frac{1}{2} |\nabla u_2|^2,$$

we have

$$\begin{aligned} \int_{\Omega} \left| \frac{\nabla(u_1 - u_2)}{2} \right|^2 &= \int_{\Omega} \frac{1}{2} |\nabla u_1|^2 + \int_{\Omega} \frac{1}{2} |\nabla u_2|^2 - \int_{\Omega} \left| \frac{\nabla(u_1 + u_2)}{2} \right|^2 \\ &\quad - \int_{\Omega} u_1 f - \int_{\Omega} u_2 f + 2 \int_{\Omega} \frac{u_1 + u_2}{2} f \\ &= J(u_1) + J(u_2) - 2J\left(\frac{u_1 + u_2}{2}\right) \leq 0 \end{aligned}$$

which implies that

$$\|\nabla(u_1 - u_2)\|_{L^2} = 0.$$

Poincaré's inequality gives

$$\|u_1 - u_2\|_{L^2} = 0 \quad \Rightarrow \quad u_1 = u_2 \text{ a.e. in } \Omega.$$

□

**Theorem 4.17.** *(Existence)  $f \in L^2(\Omega)$ , then (4.9) has a solution  $u \in H_0^1(\Omega)$ .*

*Proof.* First we prove that  $J(u)$  has a lower bound. In fact, by Young's and Poincaré's inequality,

$$J(v) = \frac{1}{2} \|\nabla v\|_{L^2}^2 - \int_{\Omega} f v \geq \frac{1}{2} \|\nabla v\|_{L^2}^2 - \frac{1}{4} \|\nabla v\|_{L^2}^2 - C \|f\|_{L^2}^2 \geq -C(d) \|f\|_{L^2}^2.$$

Let

$$m = \inf_{v \in H_0^1(\Omega)} J(v).$$

Let  $\{v_k\}_{k=1}^{\infty} \subset H_0^1(\Omega)$  be a minimizing sequence such that

$$J(v_k) \leq m + \frac{1}{k}.$$

Actually,  $\{v_k\}$  is a Cauchy sequence in  $H^1(\Omega)$  because of the following inequality which comes from similar discussions to the proof of uniqueness, for  $k, l \rightarrow \infty$ ,

$$\left\| \nabla \frac{(v_k - v_l)}{2} \right\|_{L^2}^2 = J(v_k) + J(v_l) - 2J\left(\frac{v_k + v_l}{2}\right) \leq m + \frac{1}{k} + m + \frac{1}{l} - 2m \leq \frac{1}{k} + \frac{1}{l} \rightarrow 0.$$

Due to the completeness of  $H_0^1$ , there must  $\exists u \in H_0^1(\Omega)$  such that

$$v_k \rightarrow u, \quad \text{in } H^1(\Omega).$$

Taking limit as  $k \rightarrow \infty$ , we have  $J(v_k) \rightarrow J(u)$  and  $J(u) = m$ . □

4.8. **\*\*\*Lax-Milgram theorem and existence.** We first list the Lax-Milgram theorem from functional analysis, then prove the existence of weak solution of (4.8).

**Theorem 4.18** (Lax-Milgram theorem). *Let  $H$  be a Hilbert space. Assume  $a(u, v)$  is a bi-linear mapping from  $H \times H$  to  $\mathbb{R}$  and satisfies*

- *Bounded.*  $\exists M \geq 0$  such that  $|a(u, v)| \leq M\|u\| \cdot \|v\|$ ,  $\forall u, v \in H$ .
- *Coercive.*  $\exists \delta > 0$  such that  $a(u, u) \geq \delta\|u\|^2$ ,  $\forall u \in H$ .

*Then for any bounded linear functional  $F(v)$  on  $H$ , there exists a unique  $u \in H$  such that*

$$F(v) = a(u, v), \quad \forall v \in H.$$

and

$$\|u\| \leq \frac{1}{\delta}\|F\|.$$

*Proof.* For any fixed  $u \in H$ , Riesz representation theorem implies that  $\exists Au \in H$  such that

$$a(u, v) = (Au, v), \quad \forall v \in H.$$

The linearity of  $Au$  in  $u$  is obvious due to the fact that  $a(u, v)$  is linear in  $u$ . Furthermore,

$$(Au, v) \leq M\|u\| \cdot \|v\|, \quad \Rightarrow \quad \|Au\| \leq M\|u\|.$$

Coercivity gives that  $\forall u \in H$ ,

$$\delta\|u\|^2 \leq a(u, u) = (Au, u) \leq \|Au\| \cdot \|u\|, \quad \Rightarrow \quad \|Au\| \geq \delta\|u\|.$$

Thus  $A^{-1}$  exists. We claim that  $R(A) = H$ .

First  $R(A)$  is closed. In fact, choose any Cauchy sequence  $\{Au_k\}$  in  $R(A)$ , then  $\lim_{k \rightarrow \infty} Au_k = v$ . By coercivity, we have

$$\delta\|u_k - u_l\| \leq \|Au_k - Au_l\|,$$

which means  $\{u_k\}$  is also a Cauchy sequence in  $H$ .  $\exists u \in H$  such that

$$\lim_{k \rightarrow \infty} u_k = u.$$

Thus

$$Au = \lim_{k \rightarrow \infty} Au_k = v.$$

If  $R(A) \neq H$ ,  $\exists w \neq 0$  in  $H$  such that

$$(Au, w) = 0, \quad \forall u \in H,$$

which contradicts with coercivity if we choose  $w = u$ . Thus  $R(A) = H$ .

For any linear functional  $F(v)$  on  $H$ , by Riesz representation theorem, we have a unique  $w \in H$  s.t.

$$F(v) = (w, v).$$

Let  $u = A^{-1}w$ , we have

$$\|u\| \leq \|A^{-1}\| \cdot \|w\| \leq \frac{1}{\delta} \|F\|$$

and

$$F(v) = (Au, v).$$

□

**Theorem 4.19.** For  $f \in L^2(\Omega)$ , there exists a solution  $u \in H_0^1(\Omega)$  of (4.8).

*Proof.* Let the bilinear functional be defined by

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v.$$

Then it is coercive

$$a(u, u) \geq \|\nabla u\|_{L^2}^2 \geq C\|u\|_{H^1}^2.$$

Lax-Milgram theorem implies that  $\forall f \in L^2(\Omega)$ , there exists a unique  $u \in H_0^1(\Omega)$  such that

$$a(u, v) = \langle f, v \rangle, \quad \forall v \in H_0^1(\Omega).$$

□

**4.9. Energy Estimate.** Energy methods for Poisson's equation is easy. I will not talk about it here, but leave it as an exercise. The energy estimate also shows that  $-\Delta u = f$  in  $\Omega$  and  $u = h$  on  $\partial\Omega$  has at most one solution in  $C^2(\Omega) \cap C^1(\bar{\Omega})$ .

#### 4.10. Problems.

- (1) Try to derive energy estimates for Dirichlet problem of Poisson equation.
- (2) Modify the proof of the mean value formulas to show for  $n \geq 3$  that

$$u(0) = \int_{\partial B(0,r)} g dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left( \frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f dx,$$

$$\text{provided } \begin{cases} -\Delta u = f, & x \in B(0, r) \\ u = g, & x \in \partial B(0, r) \end{cases}.$$

- (3) We say  $v \in C^2(\bar{\Omega})$  is subharmonic if  $-\Delta v \leq 0$  in  $\Omega$ .
  - (a) Prove for subharmonic  $v$  that

$$v(x) \leq \int_{B(x,r)} v dy, \quad \text{for all } B(x, r) \subset \Omega.$$

- (b) Prove that therefore  $\max_{\bar{\Omega}} v = \max_{\partial\Omega} v$ .
- (c) Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be smooth and convex. Assume  $u$  is harmonic and  $v := \phi(u)$ . Prove  $v$  is subharmonic.
- (d) Prove  $v := |Du|^2$  is subharmonic, whenever  $u$  is harmonic.

- (4) Let  $B^+(R) = \{(x, y) : x^2 + y^2 < R^2, y > 0\}$ , try to find the Green's function of the following problem

$$\begin{cases} -\Delta u = f(x, y), & (x, y) \in B^+(R), \\ u|_{\partial B^+(R) \cap \{y > 0\}} = \varphi(x, y), \\ u_y|_{y=0} = \psi(x, 0), & -R \leq x \leq R. \end{cases}$$

Furthermore, give the representation formula of solution.

- (5)  $\Omega$  is a bounded open subset of  $\mathbb{R}^n$ ,  $u(x)$  is a classical solution of

$$\begin{cases} -\Delta u + c(x)u = f(x), & x \in \Omega, \\ (\nabla u \cdot \gamma + \alpha(x)u)|_{\Gamma_1} = \varphi_1, & u|_{\Gamma_2} = \varphi_2 \end{cases}$$

where  $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ ,  $\Gamma_1 \cap \Gamma_2 = \emptyset$ ,  $\Gamma_2 \neq \emptyset$ .

If  $c(x) \geq 0$ ,  $\alpha(x) \geq \alpha_0 > 0$ , try to prove the following estimate,

$$\max_{\Omega} |u(x)| \leq C(\alpha_0, \text{diam}\Omega) \left[ \sup_{\Omega} |f| + \sup_{\Gamma_1} |\varphi_1| + \sup_{\Gamma_2} |\varphi_2| \right].$$

- (6) Try to get the Euler-Lagrange equation of the following variational problem

$$J(u) = \min_{v \in M_0} J(v), \text{ with } M_0 = \{u \in C^2(\Omega) \cap C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0\},$$

(a)  $J(v) = \int_{\Omega} \left( \frac{1}{p} |\nabla v|^p - fv \right) dx, p > 1$

(b)  $J(v) = \int_{\Omega} \left( \frac{1}{2m} |\nabla v^m|^2 - fv \right) dx, m > 0$

(c)  $j(v) = \int_{\Omega} (\sqrt{1 + |\nabla v|^2} dx + v^p) dx, p > 1$

- (7) If  $u \in H_0^1(\Omega)$  is a weak solution of

$$-\Delta u + u = f,$$

prove that  $u$  is a solution of variational problem

$$J(u) = \min_{v \in H_0^1(\Omega)} J(v),$$

where  $J(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx + \frac{1}{2} \int_{\Omega} v^2 dx - \int_{\Omega} f v dx$ .

- (8) Assume  $f \in L^2(\Omega)$ ,  $\varphi \in H^1(\Omega)$ ,  $c(x) \geq 0$  and  $c(x) \in C(\bar{\Omega})$ , prove that variational problem

$$J(u) = \min_{v \in M_{\varphi}} J(v)$$

has a unique solution in  $M_{\varphi} = \{u \in H^1(\Omega) : u - \varphi \in H_0^1(\Omega)\}$ , where

$$J(v) = \frac{1}{2} \int_{\Omega} (|\nabla v|^2 + c(x)v^2 - fv) dx.$$

Furthermore, show that the solution of variational problem is a weak solution of

$$-\Delta u + c(x)u = f \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega.$$

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