Lehrstuhl für Mathematik IV
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Introduction to PDEs

1. Assuming $u(x,t) = F(x+t) + G(x-t)$, solve the problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & (x,t) \in (0,1) \times (0,\infty) \\ u(0,t) = 0 & t > 0 \\ u(1,t) = 0 & t > 0 \\ u(x,0) = 0 & 0 < x < 1 \\ u_t(x,0) = \sin(\pi x) & 0 < x < 1. \end{cases}$$

2. (a) Use the change of variables $\xi = x+t$, $\eta = x-t$, to show that $u_{tt} - u_{xx} = -4\tilde{u}_{\xi\eta}$, where $\tilde{u}$ is the C^2 function $u(x,t)$ expressed in the ξ, η variables.

   (b) Use (a) to conclude that $u(x,t) = F(x+t) + G(x-t)$ is the general solution to $u_{tt} - u_{xx} = 0$ and proceed as in the class to derive the d'Alembert formula for the initial value problem of the wave equation in $\mathbb{R} \times (0,\infty)$ with $u(x,0) = g(x)$ in $\mathbb{R}$ and $u_t(x,0) = h(x)$ in $\mathbb{R}$.

   (c) Use (a) to derive the corresponding formula for the solution of $u_{tt} - u_{xx} = x^2 - t^2$ in $\mathbb{R} \times (0,\infty)$ with $u(x,0) = g(x)$ in $\mathbb{R}$ and $u_t(x,0) = h(x)$ in $\mathbb{R}$.

3. Given $g \in C^1(\mathbb{R})$, solve the following equation using characteristics:

$$\begin{cases} x_1 u_{x_1} + x_2 u_{x_2} = 2u & \text{in } U = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 > 1\}, \\ u(x_1, 1) = g(x_1) & \text{on } \partial U. \end{cases}$$

4. Use characteristics to produce the formula for the solution of the non-homogeneous transport equation from the first lecture; that is

$$\begin{cases} u_t + b \cdot Du = f & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

   where $b \in \mathbb{R}^n$ is fixed and $f = f(x,t) \in C\big(\mathbb{R}^n \times (0,\infty)\big)$ and $g = g(x) \in C^1(\mathbb{R}^n)$.