Lehrstuhl für Mathematik IV
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Introduction to PDEs

1. Let u be the minimizer of the following variational problem

$$J(u) = \inf_{v \in M_0} J(v), \text{ with } M_0 = \{u \in C^2(\Omega) \cap C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}.$$

Derive the Euler-Lagrange equation (the differential equation that u satisfies) for the following cases.

(a) $J(v) = \int_\Omega (\frac{1}{p}|\nabla v|^p - fv)\, dx$, $p > 1$,

(b) $J(v) = \int_\Omega (\frac{1}{2m}|\nabla v^m|^2 - fv)\, dx$, $m > 0$,

(c) $J(v) = \int_\Omega (\sqrt{1 + |\nabla v|^2} + v^p)\, dx$, $p > 1$,

Hint: consider $j(\varepsilon) = J(u + \varepsilon\varphi)$, $\forall \varphi \in M_0$ and then compute $j'(\varepsilon)|_{\varepsilon=0}$

2. Let Ω be a bounded open subset of $\mathbb{R}^n$ and

$$J(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 + |v|^2)\, dx - \int_\Omega fv\, dx.$$

Prove that $\forall f \in L^2(\Omega)$, the variational problem

$$J(u) = \inf_{v \in H_0^1} J(v)$$

has a unique solution.

3. Let Ω be a bounded open subset of $\mathbb{R}^n$. Assume that $f \in L^2$, we consider the following problem

$$\begin{aligned} -\Delta u + u &= f, \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= 0. \end{aligned}$$

(a) Give the definition of weak solution for the above problem.

(b) Use Lax-Milgram theorem to prove the existence of the weak solution you have defined.

Due to 27.11 20:00 in box 46216