Lehrstuhl für Mathematik IV
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Introduction to PDEs

1. The heat kernel $K(x,t;\xi,\tau)$ is given by

$$K(x,t;\xi,\tau) = \frac{1}{(4\pi(t-\tau))^{\frac{1}{2}}} e^{-\frac{|x-\xi|^2}{4(t-\tau)}}, \quad x,\xi \in \mathbb{R},\, t > \tau.$$

   (a) Prove that
   $$(\partial_t - \Delta_x) K(x,t;\xi,\tau) = 0, \quad \forall x,\xi \in \mathbb{R},\, t > \tau.$$

   (b) Prove that for fixed ξ and τ, $K(x,t;\xi,\tau)$ is the solution of

$$\begin{aligned} u_t - \Delta_x u &= 0, \quad x \in \mathbb{R},\, t > \tau, \\ u|_{t=\tau} &= \delta(x-\xi), \quad x \in \mathbb{R}, \end{aligned}$$

   where the initial condition is understood in the sense of distribution.

2. Give a formal derivation of the solution to the following half-line problem

$$\begin{aligned} u_t - u_{xx} &= 0, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ u|_{t=0} &= x, \quad x \in \mathbb{R}_+, \\ u_x|_{x=0} &= 0, \quad t > 0. \end{aligned}$$

   Hint: use even extension

3. Suppose that $u_0(x)$ is a bounded, continuous function in $\mathbb{R}^n$ and has compact support. Furthermore $u(x,t)$ is the classical solution of

$$\begin{aligned} u_t - \Delta u &= e^{-t}, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \\ u|_{t=0} &= u_0, \quad x \in \mathbb{R}^n. \end{aligned}$$

   Prove that for all x in $\mathbb{R}^n$, there holds

$$u(x,t) \to 1, \quad \text{as}\, t \to \infty.$$

   Hint: use the Poisson's formula

Due to 23.10 20:00 in box 46216