Lehrstuhl für Mathematik IV
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FSS 2020

Dynamische Systeme - Lösung Ü8

Aufgabe 3.) *(8 Punkte)*
Betrachten Sie das Randwerproblem

$$\begin{cases} \ddot{x} + g(x) = 0, \\ x(0) = x(1) = 0, \end{cases} \tag{0.1}$$

wobei $\ddot{x} := \frac{d^2}{dt^2}x(t)$ und $g : \mathbb{R} \to \mathbb{R}$ lokal Lipschitz seien.
Die **Shooting-Methode** zur Lösung solcher Randwerprobleme besteht drin, das entsprechende Anfangswertproblem mit $x(0) = 0, \dot{x}(0) = a$ zu lösen, und dann eine Nullstelle der Abbildung $\phi \in C^1(\mathbb{R})$ definiert durch $\phi(a) = x(1, a)$ zu suchen. Dabei bezeichnet $x(t, a)$ die Lösung dieses Anfangswertproblems.
Ihre Aufgabe: Beweisen Sie mit dieser Methode einen Existensatz im Fall eines beschränkten g, d.h. für eine Konstante M,

$$|g(x)| \leqslant M,$$

für alle $x \in \mathbb{R}$.

Lösung:

Remark: The main idea of solving the Boundary Value Problem (BVP) in (0.1) is to reduce the problem into separate Initial Value Problems (IVP) with the initial value of $x(0) = 0$ and $x'(t) = a$ and satisfy the left boundary value problem. The function a is known as a 'slope target' so that it will align the solution of IVP to the right boundary value of BVP, hence 'shooting' at the right boundary value. This is usually done by some numerical methods such as Newton's Descent (c.f. Numerical ODE course). In this question, we only concern ourselves if the BVP in (0.1) is well-posed.

1.) Linear ODE. Suppose the ODE in (0.1) is linear. Suppose for all $t \in [0, 1]$, $u(t)$ is a solution to the following IVP,

$$\begin{cases} u''(t) + g(u) = 0, \\ u(0) = 0, u'(t) = 0, \end{cases} \tag{0.2}$$

and $v(t)$ is the solution for another IVP,

$$\begin{cases} v''(t) + g(v) = 0, \\ v(0) = 0, v'(t) = a. \end{cases} \tag{0.3}$$

Now since g is locally Lipschitz, we know that both the solution above exists and unique, and that the IVP (0.2) and (0.3) are well-posed. You may show this by reducing the 2nd-order linear ODE problem into a System of 1st-order ODE.

We will show for only one IVP (0.3), as the other IVP follows the same argument: denote $w_1 = v$ and $w_2 = v'$, and then define the vector $\omega = (w_1, w_2)$, for $t \in [0, 1]$ we have the following system of 1st-order ODE,

$$\omega'(t) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}'(t) = \begin{pmatrix} v'(t) \\ v''(t) \end{pmatrix} = \begin{pmatrix} w_2 \\ -g(w_1) \end{pmatrix},$$

with initial value of $\omega(0) = (0, a)$, where $a \in \mathbb{R}$ is a constant to be determined later. Now since g is locally Lipschitz, the solution to the problem exists and is unique.

Since a linear combination of solution is a solution, then the following linear combination is the solution to the BVP in (0.1),

$$x(t) = u(t) + a \cdot v(t) =: x(t, a), \tag{0.4}$$

where the a is to be determined by matching it to the right-boundary value ('shooting' at the right boundary), i.e.,

$$0 = x(1, a) = u(1) + a \cdot v(1) \;\Leftrightarrow\; a = -\frac{u(1)}{v(1)},$$

Thus, if $v(1)$ is non-zero, then the 'slope target' a, can be determined numerically and is a constant in $\mathbb{R}$.

Therefore, given g is locally Lipschitz and linear, the solution u and v exist and are unique, therefore, by linearity argument in (0.4), the solution to the BVP problem in (0.1) x, also exists and unique. Then we are done.

2.) Non-linear ODE. For non-linear ODE, the 'slope target' a has to now satisfy a nonlinear equation of the form

$$x(1, a) = 0,$$

where $x(1, a)$ is the solution to a specified IVP problem at boundary when $t = 1$ with initial slope of a. We shall denote here $\phi(a) = x(1, a)$. Essentially, we want to find the root of ϕ by determining the a. This is done predominantly either by Newton's method. But we will not concern ourselves in finding exact solution of ϕ.

However, to use Newton method, one need to also solve the derivative of x with respect to the 'slope target' a, at each iteration. Therefore, we denote

$$z(t, a) := \frac{\partial x(t, a)}{\partial a}.$$

Observe that if we take 2nd-derivative in t on z, by Chain-rule we have

$$\begin{aligned} z''(t, a) &= \frac{\partial^2}{\partial t^2}\frac{\partial}{\partial a} x(t, a) \\ &= -\big(g_x z(t, a) + g_{x'} z'(t, a)\big). \end{aligned}$$

As in (0.3), we may have the following 2nd-order linear ODE

$$\begin{cases} z'' + g_x z + g_{x'} z' = 0, \forall t \in (0, 1) \\ z(0, a) = 0, z'(0, a) = 1, \end{cases} \tag{0.5}$$

for all $a \in \mathbb{R}$. Denote $\eta = (z, z')$, then we have reduce (0.5) to the following system of 1st-order ODE

$$\eta'(t, a) = \begin{pmatrix} 0 & 1 \\ -g_x & -g_{x'} \end{pmatrix} \cdot \eta(t, a) =: A \cdot \eta,$$

with initial value $\eta(0, a) = (0, 1)$. Then if A satisfies the usual Lipschitz condition, by Cauchy-Lipschitz Theorem, there exists a unique solution then we are done.