Isoperimetric inequality with radial densities

Georgios Psaradakis

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  Hence $Vol(E) \leq Vol(B_{r_k})\ \ \forall\ E \in \mathscr{A}_k$, where $r_k = \left(k/n\omega_n\right)^{1/(n-1)}$ and $\omega_n = Vol(B_1)$ (recall here that $Per(B_r) = n\omega_n r^{n-1} \Rightarrow Per(B_{r_k}) = n\omega_n \frac{k}{n\omega_n} = k$).

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  $Per(B_r) = n\omega_n r^{n-1} \Rightarrow Per(B_{r_k}) = n\omega_n \frac{k}{n\omega_n} = k$).
- But $Vol(B_r) = \omega_n r^n$, therefore $Vol(E) \le \omega_n (k/n\omega_n)^{n/(n-1)} \Rightarrow$
  $(Vol(E))^{(n-1)/n} \le \omega_n^{1-1/n} k/n\omega_n = k/(n\omega_n^{1/n})$. Conclusion:
  $Per(E) \ge n\omega_n^{1/n}(Vol(E))^{(n-1)/n} \ \ \forall$ bounded smooth domain $E \subset \mathbb{R}^n$.

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  Hence $Vol(E)\le Vol(B_{r_k})\ \ \forall\ E\in\mathscr{A}_k$, where $r_k=\left(k/n\omega_n\right)^{1/(n-1)}$ and $\omega_n = Vol(B_1)$ (recall here that $Per(B_r)=n\omega_n r^{n-1} \Rightarrow Per(B_{r_k})=n\omega_n\frac{k}{n\omega_n}=k$).
- But $Vol(B_r)=\omega_n r^n$, therefore $Vol(E)\le\omega_n\left(k/n\omega_n\right)^{n/(n-1)}\Rightarrow \left(Vol(E)\right)^{(n-1)/n}\le\omega_n^{1-1/n}k/n\omega_n=k/(n\omega_n^{1/n})$. Conclusion:
  $Per(E)\ge n\omega_n^{1/n}\left(Vol(E)\right)^{(n-1)/n}$ $\forall$ bounded smooth domain $E\subset\mathbb{R}^n$.
- This result is true for any $\mathscr{L}^n$-measurable set provided $Vol(E)$ is replaced by $\mathscr{L}^n(E)=\int_E 1\,\mathrm{d}\mathscr{L}^n$ and $Per(E)$ by $\mathscr{P}(E):=\sup\left\{\int_E \operatorname{div}\phi\ \mathrm{d}\mathscr{L}^n:\ \phi\in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\ |\phi|\le 1 \text{ in } \mathbb{R}^n\right\}<\infty.$

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  Hence $Vol(E) \leq Vol(B_{r_k}) \ \ \forall\ E \in \mathscr{A}_k$, where $r_k = (k/n\omega_n)^{1/(n-1)}$ and $\omega_n = Vol(B_1)$ (recall here that $Per(B_r) = n\omega_n r^{n-1} \Rightarrow Per(B_{r_k}) = n\omega_n \frac{k}{n\omega_n} = k$).
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  $Per(E) \geq n\omega_n^{1/n}(Vol(E))^{(n-1)/n}$ $\forall$ bounded smooth domain $E \subset \mathbb{R}^n$.
- This result is true for any $\mathscr{L}^n$-measurable set provided $Vol(E)$ is replaced by $\mathscr{L}^n(E) = \int_E 1 \mathrm{d}\mathscr{L}^n$ and $Per(E)$ by $\mathscr{P}(E) := \sup\left\{\int_E \mathrm{div}\phi \ \mathrm{d}\mathscr{L}^n : \ \phi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\ |\phi| \leq 1 \text{ in } \mathbb{R}^n\right\} < \infty$.
- Note that if a bounded domain E has Lipschitz boundary then $\mathscr{P}(E) = \mathscr{H}^{n-1}(\partial E)$ and if it has $\mathscr{C}^1$ boundary then $\mathscr{P}(E)$ agrees with the usual surface area $Per(E)$ of E from calculus.

Densities

- Given positive functions v, w on $\mathbb{R}^n$, we define the weighted volume and weighted perimeter of a smooth bounded domain $E \subset \mathbb{R}^n$ as
  $Vol(E;v) := \int_E v(x)\mathrm{d}x$, and

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- If E is just $\mathscr{L}^n$-measurable then we may define the above quantities as
  $\mathscr{L}^n(E; v) := \int_E v\mathrm{d}\mathscr{L}^n$,

  and (under some integrability conditions on w; see Baldi, A. *Weighted BV functions.* Houston J. Math. 2001)

  $\mathscr{P}(E; w) := \sup\left\{\int_E \mathrm{div}\phi\ \mathrm{d}x :\ \phi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n),\ |\phi| \leq w \text{ in } \mathbb{R}^n\right\} < \infty$

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- Such positive functions v, w are called densities on $\mathbb{R}^n$.
- Iso-problem with densities: for the case of a single density; i.e. $v = w$ (e.g. in the classical isoperimetric inequality we have $v = w = 1$) much is known (recent highlight: proof of Brakke's log-convex density conjecture by Chambers). The iso-problem with double density has attracted attention recently (Alvino, Betta, Brock, Cabré, Chiacchio, Csató, Howe, Mercaldo, Posteraro, Pratelli, Ros-Oton, Saracco, Serra). E.g.: the case of the two different radial densities $v(x) = \frac{1}{|x|^b}$, $w(x) = \frac{1}{|x|^a}$ with $0 > a > b - 1$ (see Csató D.I.E. 2015 and Alvino et al. JMAA 2017).

Its equivalent functional formulation and its weighted generalization

- In 1960 Federer and Fleming and at the same time Maz'ya showed that it is equivalent with the sharp form of an inequality originally due to Gagliardo and Nirenberg:
$\int_{\mathbb{R}^n} |\nabla f| \, \mathrm{d}x \geq n\omega_n^{1/n} \Big(\int_{\mathbb{R}^n} |f|^{n/(n-1)} \, \mathrm{d}x \Big)^{(n-1)/n} \quad \forall \ f \in C_c^1(\mathbb{R}^n).$

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- In 2007 Maz'ya and Shaposhnikova proved that if $0 \leq a < n-1$ and $an/(n-1) \leq b \leq a+1$, then
  $$\int_{\mathbb{R}^n} \frac{|\nabla f|}{|x|^a}\,\mathrm{d}x \geq \mathscr{C}_{n,a,b}\Big(\int_{\mathbb{R}^n} \frac{|f|^{(n-b)/(n-1-a)}}{|x|^b}\,\mathrm{d}x\Big)^{(n-1-a)/(n-b)} \quad \forall\ f \in C_c^1(\mathbb{R}^n),$$
  $\mathscr{C}_{n,a,b} := \big(n\omega_n(n-b)^{(n-1-a)/(1+a-b)}\big)^{(1+a-b)/(n-b)}$ is sharp.

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$\mathscr{C}_{n,a,b} := \big(n\omega_n(n-b)^{(n-1-a)/(1+a-b)}\big)^{(1+a-b)/(n-b)}$ is sharp.
- Taking $a = b = 0$ we recover the sharp GN.
- Correspondingly, this turns out to be equivalent with a *weighted isoperimetric inequality*: for any $\mathscr{L}^n$-measurable set $E \subset \mathbb{R}^n$ there holds
$$\mathscr{P}(E;|x|^{-a}) \geq \mathscr{C}_{n,a,b}\big(\mathscr{L}^n(E;|x|^{-b})\big)^{(n-1-a)/(n-b)},$$
$$\mathscr{P}(E;|x|^{-a}) := \sup\Big\{\int_E \mathrm{div}\phi \,\mathrm{d}x :\ \phi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\ |\phi| \leq |x|^{-a} \text{ in } \mathbb{R}^n\Big\} < \infty$$
and $\mathscr{L}^n(E;|x|^{-b}) := \int_E |x|^{-b}\mathrm{d}x$.

A substitute for the case $a = n-1$, $b = n$

- What is going wrong with the end point case $a = n-1$? The assumptions for the parameter b gives $b = n$ in this case and so the right hand side is infinite unless f is supported away from the origin.

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- Set $X(t) := (1-\log t)^{-1}$, $t \in (0,1]$, and observe $\lim_{t\to 0^+} X(t) = 0$. A direct computation shows that given $R > 0$, for any $\delta \in (0,R]$ one has $\int_{B_\delta(0)} |x|^{-n} X^{1+\theta}\left(\frac{|x|}{R}\right) \mathrm{d}x < \infty \Leftrightarrow \theta > 0$.

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- Theorem Suppose Ω is a bounded domain in $\mathbb{R}^n$, containing the origin and set $R_\Omega := \sup_{x\in\Omega}|x|$. For all $\gamma \in (0, n-1]$ and any $f \in C_c^1(\Omega)$, it holds that

$$\int_\Omega \frac{|\nabla f|}{|x|^{n-1}} X^\gamma\left(\frac{|x|}{R_\Omega}\right) \mathrm{d}x \ge \mathfrak{C}_{n,\gamma}\left(\int_\Omega \frac{|f|^{n/(n-1)}}{|x|^n} X^{1+\gamma n/(n-1)}\left(\frac{|x|}{R_\Omega}\right) \mathrm{d}x\right)^{1-1/n},$$

where $\mathfrak{C}_{n,\gamma} := n\omega_n^{1/n}(\gamma/(n-1))^{1-1/n}$ is sharp.

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- Whenever $E \subseteq \Omega$ is $\mathscr{L}^n$-measurable, recall once more that we set $\mathscr{L}^n(E; |x|^{-n} X^{1+\gamma n/(n-1)}(|x|/R_\Omega)) := \int_E |x|^{-n} X^{1+\gamma n/(n-1)}(|x|/R_\Omega)\,\mathrm{d}x$,
  $\mathscr{P}(E; |x|^{1-n} X^{\gamma}(|x|/R_\Omega)) :=$
  $\sup\Big\{\int_E \operatorname{div}\varphi\,\mathrm{d}x : \ \varphi \in C_c^1(\Omega;\mathbb{R}^n),\ |\varphi(x)| \le |x|^{1-n} X^{\gamma}(|x|/R_\Omega) \text{ in } \Omega\Big\} < \infty.$
  Theorem For all $\gamma \in (0, n-1]$ there holds

  $$\mathscr{P}(E; |x|^{1-n} X^{\gamma}(|x|/R_\Omega)) \ge \mathfrak{C}_{n,\gamma}\Big(\mathscr{L}^n(E; |x|^{-n} X^{1+\gamma n/(n-1)}(|x|/R_\Omega))\Big)^{1-1/n},$$

  for any $\mathscr{L}^n$-measurable set $E \subseteq \Omega$ with $\mathscr{P}(E; |x|^{1-n} X^{\gamma}(|x|/R_\Omega)) < \infty$.
  Moreover, equality holds if E is a ball centered at the origin.

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  Theorem For all $\gamma\in(0,n-1]$ there holds
  $$\mathscr{P}(E;|x|^{1-n}X^{\gamma}(|x|/R_\Omega))\ge\mathfrak{C}_{n,\gamma}\Big(\mathscr{L}^n(E;|x|^{-n}X^{1+\gamma n/(n-1)}(|x|/R_\Omega))\Big)^{1-1/n},$$
  for any $\mathscr{L}^n$-measurable set $E\subseteq\Omega$ with $\mathscr{P}(E;|x|^{1-n}X^{\gamma}(|x|/R_\Omega))<\infty$. Moreover, equality holds if E is a ball centered at the origin.
- From now on we write $w(x)=|x|^{1-n}X^{\gamma}(|x|/R_\Omega)$ and $v(x)=|x|^{-n}X^{1+\gamma n/(n-1)}(|x|/R_\Omega)$.

proof of the inequality

- Consider the best constant in $\mathbb{B}^n := B_1(0)$

$$\mathfrak{C} := \inf_{\substack{f \in C_c^1(\mathbb{B}^n) \\ f \neq 0}} \frac{\int_{\mathbb{B}^n} |\nabla f| w \, \mathrm{d}x}{\left(\int_{\mathbb{B}^n} |f|^{n/(n-1)} v \, \mathrm{d}x \right)^{1-1/n}}.$$

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- Change variables by $f(x) = h(\tau, \theta)$, $\tau := X^{-\gamma}(|x|)$, $\theta := x/|x|$, to get

$$\mathfrak{C} = \gamma^{1-1/n} \inf_{h \in \mathscr{A}} \frac{\int_1^\infty \int_{\mathbb{S}^{n-1}} \tau^{-1} \left(h_\tau^2 + \left(\gamma \tau^{1-1/\gamma}\right)^{-2} |\nabla_\theta h|^2 \right)^{1/2} \, \mathrm{d}S(\theta) \mathrm{d}\tau}{\left(\int_1^\infty \int_{\mathbb{S}^{n-1}} \tau^{-1-n/(n-1)} |h|^{n/(n-1)} \, \mathrm{d}S(\theta) \mathrm{d}\tau \right)^{1-1/n}},$$

where $\mathscr{A} := \{ g \in C^1([1,\infty) \times \mathbb{S}^{n-1}) \setminus \{0\} : \; g(1,\theta) = 0 \}$.

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- The sharp GN inequality in $\mathbb{B}^n$ is

$$n\omega_n^{1/n} = \inf_{\substack{f\in C_c^1(\mathbb{B}^n)\\ f\not\equiv 0}} \frac{\int_{\mathbb{B}^n} |\nabla f| \,\mathrm{d}x}{\left(\int_{\mathbb{B}^n} |f|^{n/(n-1)} \,\mathrm{d}x\right)^{1-1/n}}.$$

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- To compare the two infima we observe that since $\gamma \in (0, n-1]$ and $\tau \geq 1$, we know
$$\gamma \tau^{1-1/\gamma} \leq (n-1)\tau.$$
Hence we may combine these equations to conclude
$$\mathfrak{C} \geq n\omega_n^{1/n}\left(\frac{\gamma}{n-1}\right)^{1-1/n}.$$

- Theorem If $1 \le p < n$, then for all $\alpha \in (1-p, n+1-2p]$ and any $f \in C_c^1(\Omega)$, it holds that
$\int_\Omega \frac{|\nabla f|^p}{|x|^{n-p}} X^\alpha\left(\frac{|x|}{R_\Omega}\right)\,\mathrm{d}x \ge$
$C_{n,\alpha,p}^p\left(\int_\Omega \frac{|f|^{np/(n-p)}}{|x|^n} X^{1+(\alpha+p-1)n/(n-p)}\left(\frac{|x|}{R_\Omega}\right)\,\mathrm{d}x\right)^{1-p/n}$,
with the sharp constant $C_{n,\alpha,p} = \left(\frac{\alpha+p-1}{n-p}\right)^{1-1/n} S_{n,p}$, where $S_{n,p}$ is the sharp constant in the Sobolev inequality:
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$C_{n,\alpha,p}^p \left(\int_\Omega \frac{|f|^{np/(n-p)}}{|x|^n} X^{1+(\alpha+p-1)n/(n-p)}\left(\frac{|x|}{R_\Omega}\right) \mathrm{d}x\right)^{1-p/n}$,
with the sharp constant $C_{n,\alpha,p} = \left(\frac{\alpha+p-1}{n-p}\right)^{1-1/n} S_{n,p}$, where $S_{n,p}$ is the sharp constant in the Sobolev inequality:
$\int_\Omega |\nabla f|^p \,\mathrm{d}x \ge S_{n,p}\left(\int_\Omega |f|^{np/(n-p)} \,\mathrm{d}x\right)^{1-p/n}$.
- Theorem Let $f \in C_c^1(\Omega)$, $p \ge 1$ and $\gamma \in (0, n-1]$. Then
$$\int_\Omega \frac{|\nabla f|^p}{|x|^{n-p}} X^\vartheta\left(\frac{|x|}{R_\Omega}\right) \mathrm{d}x \ge \int_{\Omega^\star} \frac{|\nabla f^\star|^p}{|x|^{n-p}} X^\vartheta\left(\frac{|x|}{R_\Omega}\right) \mathrm{d}x,$$
where ϑ is given by $\vartheta := \gamma p - (1+\gamma n/(n-1))(p-1)$. Here:
-$E^\star \subset \mathbb{R}^n$ is the ball centered at 0 and s.t. $\mathscr{L}^n(E; v) = \mathscr{L}^n(E^\star; v)$,
-$\chi_E^\star : \mathbb{R}^n \mapsto \{0,1\}$ is the characteristic function of $E^\star$,
-$f^\star(x) := \int_0^\infty \chi^\star_{\{|f|>t\}}(x)\mathrm{d}t$.

- **Theorem** If $1 \le p < n$, then for all $\alpha \in (1-p, n+1-2p]$ and any $f \in C_c^1(\Omega)$, it holds that
  $\int_\Omega \frac{|\nabla f|^p}{|x|^{n-p}} X^\alpha\left(\frac{|x|}{R_\Omega}\right) \mathrm{d}x \ge$
  $C_{n,\alpha,p}^p\left(\int_\Omega \frac{|f|^{np/(n-p)}}{|x|^n} X^{1+(\alpha+p-1)n/(n-p)}\left(\frac{|x|}{R_\Omega}\right) \mathrm{d}x\right)^{1-p/n}$,
  with the sharp constant $C_{n,\alpha,p} = \left(\frac{\alpha+p-1}{n-p}\right)^{1-1/n} S_{n,p}$, where $S_{n,p}$ is the sharp constant in the Sobolev inequality:
  $\int_\Omega |\nabla f|^p \, \mathrm{d}x \ge S_{n,p}\left(\int_\Omega |f|^{np/(n-p)} \, \mathrm{d}x\right)^{1-p/n}$.
- **Theorem** Let $f \in C_c^1(\Omega)$, $p \ge 1$ and $\gamma \in (0, n-1]$. Then
  $$\int_\Omega \frac{|\nabla f|^p}{|x|^{n-p}} X^\vartheta\left(\frac{|x|}{R_\Omega}\right) \mathrm{d}x \ge \int_{\Omega^\star} \frac{|\nabla f^\star|^p}{|x|^{n-p}} X^\vartheta\left(\frac{|x|}{R_\Omega}\right) \mathrm{d}x,$$
  where ϑ is given by $\vartheta := \gamma p - (1 + \gamma n/(n-1))(p-1)$. Here:
  -$E^\star \subset \mathbb{R}^n$ is the ball centered at 0 and s.t. $\mathscr{L}^n(E; v) = \mathscr{L}^n(E^\star; v)$,
  -$\chi_E^\star : \mathbb{R}^n \mapsto \{0, 1\}$ is the characteristic function of $E^\star$,
  -$f^\star(x) := \int_0^\infty \chi^\star_{\{|f|>t\}}(x) \mathrm{d}t$.
- Note that
  $\mathscr{P}(E; w) \ge \mathfrak{C}_{n,\gamma}\left(\mathscr{L}^n(E; v)\right)^{1-1/n} = \mathfrak{C}_{n,\gamma}\left(\mathscr{L}^n(E^\star; v)\right)^{1-1/n} = \mathscr{P}(E^\star; w)$.

- Theorem Let Ω be a bounded domain in $\mathbb{R}^n$ containing the origin, and set $R_\Omega := \sup_{x\in\Omega}|x|$. Then for all $\gamma > 0$, $s \geq n$, and any $f \in C_c^1(\Omega\setminus\{0\})$, it holds that
$$\int_\Omega \frac{|\nabla f|}{|x|^{s-1}}\,\mathrm{d}x - (s-n)\int_\Omega \frac{|f|}{|x|^s}\,\mathrm{d}x$$
$$\geq \frac{\gamma}{R_\Omega^{s-n}}\int_\Omega \frac{|f|}{|x|^n} X^{1+\gamma}\left(\frac{|x|}{R_\Omega}\right)\mathrm{d}x + \frac{\mathfrak{C}_{n,\gamma}}{R_\Omega^{s-n}}\left(\int_\Omega \frac{|f|^{n/(n-1)}}{|x|^n} X^{1+\gamma n/(n-1)}\left(\frac{|x|}{R_\Omega}\right)\mathrm{d}x\right)^{1-1/n},$$
where the second term as well as the first term on the right fail to appear when $\gamma = 0$.

The End