Lehrstuhl für Mathematik IV
Dr Georgios Psaradakis

Universität Mannheim
Fakultät für Wirtschaftsinformatik
und Wirtschaftsmathematik

Tutor: Matthew Liew

Monday 28/09/2020

Advanced analysis - HWS 2020/2021

Homework assignment #01

For the first three exercises, X stands for a nonempty set.

Exercise 1 Show that Dirac's delta set-function with mass concentrated at $x \in X$; that is

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \qquad \text{for any } A \subseteq X,$$

is a measure. [2pt]

Exercise 2 (first Borel-Cantelli lemma) Let μ be a measure on X and suppose that $\{A_k \subseteq X\}_{k\in\mathbb{N}}$ satisfies $\sum_{k\in\mathbb{N}} \mu(A_k) < \infty$. Prove that $\mu(\limsup_{k\to\infty} A_k) = 0$ (see the back page for definitions). [2pt]

Exercise 3 Define the following function on subsets A of X

$$\mu(A) = \begin{cases} 1 & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset. \end{cases}$$

Prove that μ is a measure on X and find all μ-measurable subsets of X. [2pt]

Exercise 4 Prove that $\mathscr{L}^n(I) = \mathrm{vol}(I)$ for any n-dimensional interval I. [2pt]

The following two exercises are taken from undergraduate analysis and are asked here for future use.

Exercise 5 Prove that the distance of a compact set $K \subset \mathbb{R}^n$ to a disjoint closed set $F \subset \mathbb{R}^n$ is positive. [1pt]

Exercise 6 For compact $K \subset \mathbb{R}^n$, show there exists a non-increasing sequence $\{F_k \supseteq K\}_{k\in\mathbb{N}}$ with $K = \cap_{k\in\mathbb{N}} F_k$ and each of F_k consists of a finite union of non-overlapping intervals. [1pt]

Upload the solutions on ILIAS by 12/10/20 before 12:00.

Abstract sets and sequences of abstract sets

Let X be a nonempty set and 2^X be the set of all subsets of X.

1. Writing A^c for the *complement* of $A \subseteq X$, that is $A^c := X \setminus A$, there holds

$$\Big(\bigcup_{A\in\mathscr{F}} A\Big)^c = \bigcap_{A\in\mathscr{F}} A^c \quad \text{and} \quad \Big(\bigcap_{A\in\mathscr{F}} A\Big)^c = \bigcup_{A\in\mathscr{F}} A^c,$$

for any family of sets $\mathscr{F} \subseteq 2^X$.

2. A sequence $\{A_k \subseteq X\}_{k\in\mathbb{N}}$ is said to *non-decrease* (to $\bigcup_{k\in\mathbb{N}} A_k$) if $A_k \subseteq A_{k+1}$ for all $k \in \mathbb{N}$, and to *non-increase* (to $\bigcap_{k\in\mathbb{N}} A_k$) if $A_k \supseteq A_{k+1}$ for all $k \in \mathbb{N}$.

3. For a sequence $\{A_k \subseteq X\}_{k\in\mathbb{N}}$ and $j \in \mathbb{N}$, let

$$\overline{M}_j := \bigcup_{k\geq j} A_k \quad \text{and} \quad \underline{M}_j := \bigcap_{k\geq j} A_k.$$

Then $\{\overline{M}_j\}_{j\in\mathbb{N}}$ is non-increasing, $\{\underline{M}_j\}_{j\in\mathbb{N}}$ is non-decreasing and we set

$$\limsup_{k\to\infty} A_k := \bigcap_{j\in\mathbb{N}} \overline{M}_j \quad \text{and} \quad \liminf_{k\to\infty} A_k := \bigcup_{j\in\mathbb{N}} \underline{M}_j.$$

Lehrstuhl für Mathematik IV
Dr Georgios Psaradakis

Universität Mannheim
Fakultät für Wirtschaftsinformatik
und Wirtschaftsmathematik

Tutor: Matthew Liew

Monday 12/10/2020

Advanced analysis

Homework assignment #02

Exercise 1 Let μ be a regular measure on X. Prove that if $A_k \subseteq A_{k+1}$ for all $k \in \mathbb{N}$, then $\lim_{k\to\infty} \mu(A_k) = \mu(\cup_{k\in\mathbb{N}} A_k)$. [2pt]

Exercise 2: A set $A \subseteq X$ is called σ*-finite* with respect to μ if

$$A = \cup_{k\in\mathbb{N}} A_k, \quad \text{where } A_k \text{ are } \mu\text{-measurable and } \mu(A_k) < \infty \ \forall\, k \in \mathbb{N}.$$

Prove that if $A \subseteq X$ is σ-finite with respect to μ, then there exist *disjoint* μ-measurable sets B_k, $k \in \mathbb{N}$, such that $A = \cup_{k\in\mathbb{N}} B_k$ and $\mu(B_k) < \infty$ for all $k \in \mathbb{N}$. [2pt]

Exercise 3 [**Lebesgue measure is a Radon measure**] In the lectures we have seen by applying Caratheodory's criterion that the Lebesgue measure on $\mathbb{R}^n$ is a Borel measure.

(i) Prove it is finite on compact subsets of $\mathbb{R}^n$. [2pt]

(iii) Prove that $\mathscr{L}^n(A) = \inf\{\mathscr{L}^n(G) \mid \text{open } G \supseteq A\}$. [2pt]

(iii) Use (ii) to prove it is Borel regular (i.e. $\forall\, A \subset \mathbb{R}^n$, $\exists$ Borel $B \supseteq A$ with $\mathscr{L}^n(B) \leq \mathscr{L}^n(A)$).

[2pt]

Upload the solutions on ILIAS by 19/10/20 before 12:00.

Lehrstuhl für Mathematik IV
Dr Georgios Psaradakis

Universität Mannheim
Fakultät für Wirtschaftsinformatik
und Wirtschaftsmathematik

Tutor: Matthew Liew

Monday 19/10/2020

Advanced analysis

Homework assignment #03

Exercise 1 Let $f, g : \mathbb{R}^n \to \mathbb{R}^n$ be Borel measurable. Show that $f \circ g$ is Borel measurable. [3pt]

Exercise 2: Let μ be a measure on the set $X \neq \emptyset$. A set $A \in 2^X$ is called an *atom* for μ if $0 < \mu(A) < \infty$ and for any $F \subseteq A$ we have either $\mu(F) = 0$, or $\mu(A \setminus F) = 0$. In particular, for any μ-measurable $F \subseteq A$ we have then either $\mu(F) = 0$, or $\mu(F) = \mu(A)$. For example let X be any countable set and take μ to be the counting measure on X; singletons $A = \{x\}$, $x \in X$, are atoms for μ.

Show the Lebesgue measure $\mathscr{L}^n$ has no atoms by proving that given $A \subset \mathbb{R}^n$ with $0 < \mathscr{L}^n(A) < \infty$ and $0 < \beta < \mathscr{L}^n(A)$, there is a set $B \subseteq A$ with $\mathscr{L}^n(B) = \beta$. [3pt]

[Hint: You can prove the function $f : [0, \infty] \to [0, \mathscr{L}^n(A)]$ given by $f(r) := \mathscr{L}^n\big(A \cap B_r(0)\big)$ is continuous.]

Exercise 3 Let $f : [0, \infty) \to [0, \infty)$ be concave; that is, for any $\lambda \in [0, 1]$ and all $x, y \in [0, \infty)$ there holds

$$f\big(\lambda x + (1-\lambda)y\big) \geq \lambda f(x) + (1-\lambda) f(y).$$

Prove

(i) $g : (0, \infty) \to [0, \infty)$ defined by $g(x) = f(x)/x$ is decreasing. [2pt]

(ii) f is *sub-additive*; that is $f(x+y) \leq f(x) + f(y)$ for all $x, y \geq 0$. [2pt]

Upload the solutions on ILIAS by 26/10/20 before 12:00.

Lehrstuhl für Mathematik IV
Dr Georgios Psaradakis

Universität Mannheim
Fakultät für Wirtschaftsinformatik
und Wirtschaftsmathematik

Tutor: Matthew Liew

Monday 26/10/2020

Advanced analysis

Homework assignment #04

Exercise 1 Let $E \subset \mathbb{R}^n$ be $\mathscr{L}^n$-measurable with $\mathscr{L}^n(E) < \infty$ and let $f : E \to \mathbb{R}$ be an $\mathscr{L}^n$-measurable function. Define $\mu_{E,f} : \mathbb{R} \to [0,\infty)$ by

$$\mu_{E,f}(s) := \mathscr{L}^n(\{x \in E \mid f(x) > s\}).$$

Prove the following assertions

(i) $\mu_{E,f}$ is bounded, non-increasing, $\mu_{E,f}(-\infty) = \mathscr{L}^n(E)$ and $\mu_{E,f}(\infty) = 0$. [1pt]

(ii) $\mu_{E,f}(s) - \mu_{E,f}(t) = \mathscr{L}^n(\{x \in E \mid s < f(x) \leq t\})$ whenever $s < t$. [1pt]

(iii) $\lim_{s\to t^+} \mu_{E,f}(s) = \mu_{E,f}(t)$ (thus $\mu_{E,f}$ is right-continuous). [1pt]

(iv) $\lim_{s\to t^-} \mu_{E,f}(s) = \mathscr{L}^n(\{x \in E \mid f(x) \geq t\}) = \mu_{E,f}(t) + \mathscr{L}^n(\{x \in E \mid f(x) = t\})$ (thus $\mu_{E,f}$ is continuous at $t \in \mathbb{R}$ if and only if $\mathscr{L}^n(\{x \in E \mid f(x) = t\}) = 0$). [1pt]

(v) $\mu_{E,f}$ is constant in the interval (s,t) if and only if $\mathscr{L}^n(\{x \in E \mid s < f(x) < t\}) = 0$. [1pt]

Exercise 2: Let $f,\ f_k : X \to \mathbb{R}^n$, $k \in \mathbb{N}$, be μ-summable, such that

$$\lim_{k\to\infty} \int_X |f_k - f| \mathrm{d}\mu = 0.$$

Prove that there exists a subsequence $\{f_{l_k}\}_{k\in\mathbb{N}}$ such that $f_{l_k} \to f$ μ-a.e. in X.

[Hint: Setting $\alpha_k := \int_X |f_k - f| \mathrm{d}\mu$, extract first a subsequence $\{\alpha_{\ell_k}\}_{k\in\mathbb{N}}$ such that $\sum_{k\in\mathbb{N}} \alpha_{\ell_k} \leq 1$. Apply then the monotone convergence theorem to the sequence $g_k := \sum_{j=1}^k |f_{\ell_j} - f|$.] [3pt]

Exercise 3 Prove that if $f : X \to \mathbb{R}$ is μ-summable then

$$\mu(\{x \in X \mid |f(x)| \geq s\}) \leq \frac{1}{s} \int_X |f| \, \mathrm{d}\mu \quad \text{whenever } s > 0.$$

Use this to prove that $\int_X |f| \mathrm{d}\mu = 0$ if and only if $f = 0$ μ-a.e. in X. [2pt]

Upload the solutions on ILIAS by 02/11/20 before 12:00.

Lehrstuhl für Mathematik IV
Dr Georgios Psaradakis

Universität Mannheim
Fakultät für Wirtschaftsinformatik
und Wirtschaftsmathematik

Tutor: Matthew Liew

Monday 02/11/2020

Advanced analysis

Homework assignment #05

Exercise 1 Let μ be a measure on the set X and $\mu(X) = 1$. Let $u : X \to [0, \infty]$ be μ-summable.

(i) If $\int_X u \, \mathrm{d}\mu = 1$, prove that

$$\int_X u \log u \, \mathrm{d}\mu \geq 0.$$

[2pt]

(ii) Prove that for all $p \geq 1$ there holds

$$\frac{1}{\int_X \frac{1}{u^p} \, \mathrm{d}\mu} \leq \Big(\int_X u \, \mathrm{d}\mu \Big)^p.$$

[2pt]

(iii) Prove that

$$\sqrt{1 + \Big(\int_X u \, \mathrm{d}\mu \Big)^2} \leq \int_X \sqrt{1 + u^2} \, \mathrm{d}\mu \leq 1 + \int_X u \, \mathrm{d}\mu.$$

[2pt]

Exercise 2: Let $\alpha > 0$. Given a summable function $f : (0, \alpha) \to \bar{\mathbb{R}}$, define $g : (0, \alpha) \to \bar{\mathbb{R}}$ by

$$g(x) := \int_x^\alpha \frac{f(t)}{t} \, \mathrm{d}t.$$

Prove that g is also a summable and satisfies

$$\int_0^\alpha g(x) \, \mathrm{d}x = \int_0^\alpha f(x) \, \mathrm{d}x.$$

[2pt]

Exercise 3 Compute the following limit

$$\lim_{k \to \infty} \int_0^\infty \frac{\sin(x/k)}{(1 + x/k)^k} \, \mathrm{d}x.$$

[1pt]

Exercise 4 Prove that

$$\lim_{k \to \infty} \int_0^1 \Big(1 - e^{-x^2/k} \Big) x^{-1/2} \, \mathrm{d}x = 0.$$

[1pt]

Upload the solutions on ILIAS by 09/11/20 before 12:00.

Lehrstuhl für Mathematik IV
Dr Georgios Psaradakis

Universität Mannheim
Fakultät für Wirtschaftsinformatik
und Wirtschaftsmathematik

Tutor: Matthew Liew

Monday 09/11/2020

Advanced analysis

Homework assignment #06

Exercise 1 Let μ be a measure on X. Prove the *interpolation inequality* for L^p-norms: If $1 \leq p \leq q \leq r \leq \infty$, $p \neq r$, and $f \in L^p(X,\mu) \cap L^r(X,\mu)$ then

$$\|f\|_{L^q(X,\mu)} \leq \|f\|^{\theta}_{L^p(X,\mu)} \|f\|^{1-\theta}_{L^r(X,\mu)}, \quad \text{where } \theta = \frac{1/q - 1/r}{1/p - 1/r}.$$

[4pt]

Exercise 2: (a generalization of Hölders inequality): Let μ be a measure on X. Let $k \in \mathbb{N} \setminus \{1\}$ and $p_i \in [1,\infty]$, $i = 1,\dots,k$ be such that $\sum_{i=1}^{k}(1/p_i) = 1$. Prove that given $f_i \in L^{p_i}(X,\mu)$, $i = 1,\dots,k$, we have

$$\int_X \prod_{i=1}^{k} |f_i| \mathrm{d}\mu \leq \prod_{i=1}^{k} \|f_i\|_{L^{p_i}(X,\mu)}.$$

[3pt]

Exercise 3 Let μ be a measure on X and $0 < \mu(X) < \infty$. Prove the function $A : (0,\infty] \to [0,\infty]$ given by

$$A(p) := \Big(\frac{1}{\mu(X)} \int_X |f|^p \,\mathrm{d}\mu\Big)^{1/p},$$

is nondecreasing; that is, if $0 < p_1 < p_2 \Rightarrow A(p_1) \leq A(p_2)$. [3pt]

Upload the solutions on ILIAS by 16/11/20 before 12:00.

Lehrstuhl für Mathematik IV
Dr Georgios Psaradakis

Universität Mannheim
Fakultät für Wirtschaftsinformatik
und Wirtschaftsmathematik

Tutor: Matthew Liew

Monday 16/11/2020

Advanced analysis

Homework assignment #07 - Convergence in Measure[1]

Let μ be a measure on a set X and suppose that $\mu(X) < \infty$.

Definition: We say that a sequence $\{f_k : X \to \bar{\mathbb{R}}\}_{k\in\mathbb{N}}$ of μ-measurable functions *converges in measure* to $f : X \to \bar{\mathbb{R}}$, whenever

$$\lim_{k\to\infty} \mu\big(\{x \in X \mid |f_k(x) - f(x)| > \varepsilon\}\big) = 0 \quad \forall\, \varepsilon > 0.$$

We write this as $f_k \to^{\mu} f$.

Exercise 1 Prove that if $f_k \to f$ μ-a.e. in X, then $f_k \to^{\mu} f$.

[Hint: Use Egorov's theorem.] [3pt]

Exercise 2 Prove that if $f_k \to f$ in L^p, $p \geq 1$, then $f_k \to^{\mu} f$. [3pt]

Exercise 3 Prove that if $f_k \to^{\mu} f$, then there exists a subsequence $\{f_{l_k}\}_{k\in\mathbb{N}}$ such that $f_{l_k} \to f$ μ-a.e. in X.

Exercise 4 Let $f_k \to^{\mu} f$ and $g : X \to [0,\infty]$, $g \in L^p$ with $p \geq 1$, be such that $|f_k| \leq g$ μ-a.e. in X, for all $k \in \mathbb{N}$. Show that $f_k \to f$ in L^p. [2pt]

[Hint: Split $\|f_k - f\|_p^p$ in two integrals on suitable disjoint sets. In one of them you can use Vitali's theorem on the absolute continuity of the integral. You might want to use Exercise 3 to show that $f \in L^p$.] [2pt]

Upload the solutions on ILIAS by 23/11/20 before 12:00.

[1] corrected some misprints including an exchange of the numbering of exercises 3, 4. I have enlarged the hint in 4!

Solution of Exercise 3: Let $j \in \mathbb{N}$. Taking $\varepsilon = 1/j$ in the definition of convergence in measure,

$$\lim_{k\to\infty} \mu\big(\{x \in X \mid |f_k(x) - f(x)| > 1/j\}\big) = 0.$$

Analyzing the limit,

$$\forall\, \varepsilon > 0,\ \exists\, k_\varepsilon \in \mathbb{N} \text{ such that } \mu\big(\{x \in X \mid |f_k(x) - f(x)| > 1/j\}\big) \le \varepsilon \ \ \forall\, k \ge k_\varepsilon.$$

By taking $\varepsilon = 2^{-j}$,

$$\exists\, k_j \in \mathbb{N} \text{ such that } \mu\big(\{x \in X \mid |f_k(x) - f(x)| > 1/j\}\big) \le 2^{-j} \ \ \forall\, k \ge k_j.$$

For $k = k_j$,

$$\mu\big(\{x \in X \mid |f_{k_j}(x) - f(x)| > 1/j\}\big) \le 2^{-j}.$$

Summarizing,

$$\forall\, j \in \mathbb{N},\ \exists\, k_j \in \mathbb{N} \text{ such that for } N_j := \{x \in X \mid |f_{k_j}(x) - f(x)| > 1/j\} \text{ we have } \mu\big(N_j\big) \le 2^{-j}.$$

This implies $\sum_{j\in\mathbb{N}} \mu(N_j) = 1 < \infty$ and by the Borel-Cantelli lemma (Exercise 2, Assignment #01) we get

$$\mu\big(\limsup_{j\to\infty} N_j\big) = 0. \tag{1}$$

On the other side, if $x \in N_j^c$ we have $|f_{k_j}(x) - f(x)| \le 1/j$. Hence, given $m \in \mathbb{N}$ we know that $x \in Z_m := \big(\cup_{j\ge m} N_j\big)^c$ implies $|f_{k_j}(x) - f(x)| \le 1/j$ for all $j \ge m$, hence $\lim_{j\to\infty} f_{k_j}(x) = f(x)$ for all $x \in Z_m$. Since this is true for all $m \in \mathbb{N}$, we conclude

$$\lim_{j\to\infty} f_{k_j}(x) = f(x) \text{ for all } x \in \bigcup_{m\in\mathbb{N}} Z_m. \tag{2}$$

But $\cup_{m\in\mathbb{N}} Z_m = \cup_{m\in\mathbb{N}}\big(\cup_{j\ge m} N_j\big)^c = \big(\cap_{m\in\mathbb{N}}\big(\cup_{j\ge m} N_j\big)\big)^c = \big(\limsup_{j\to\infty} N_j\big)^c$ and so (1) and (2) yield

$$\lim_{j\to\infty} f_{k_j}(x) = f(x) \ \ \mu\text{-a.e. } x \in X.$$

Remark on Exercise 4: Suppose $f_k \to f$ μ-a.e. in X and $g : X \to [0,\infty]$, $g \in L^p$ with $p \ge 1$, is such that $|f_k| \le g$ μ-a.e. in X, for all $k \in \mathbb{N}$. Then, as we have seen many times, Fatou's Lemma readily gives $f \in L^p$. Replacing a.e. convergence by convergence in measure, this remains true![2] Indeed, applying Exercise 3 we obtain a subsequence $\{f_{l_k}\}_{k\in\mathbb{N}}$ converging to f in the a.e. sense. Hence, by Fatou's lemma,

$$\int_X |f|^p \mathrm{d}\mu = \int_X \lim_{k\to\infty} |f_{l_k}|^p \mathrm{d}\mu \le \liminf_{k\to\infty} \int_X |f_{l_k}|^p \mathrm{d}\mu \le \int_X |g|^p \mathrm{d}\mu < \infty.$$

[2]More generally, one can prove that given a sequence $\{g_k : X \to [0,\infty]\}_{k\in\mathbb{N}}$ of μ-measurable functions such that $g_k \to^\mu g$ in X, then $\int_X g \mathrm{d}\mu \le \liminf_{k\to\infty} \int_X g_k \mathrm{d}\mu$. To this end, pick first a subsequence $\{g_{l_k}\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \int_X g_{l_k} \mathrm{d}\mu = \liminf_{k\to\infty} \int_X g_k \mathrm{d}\mu$, and then we may apply the above argument for $\{g_{l_k}\}_{k\in\mathbb{N}}$ as soon as we observe $g_{l_k} \to^\mu g$ too. Thus there exists a further subsequence $g_{m_{l_k}}$, such that $g_{m_{l_k}} \to g$ μ-a.e. in X. So by Fatou's lemma,

$$\int_X g \mathrm{d}\mu = \int_X \lim_{k\to\infty} g_{m_{l_k}} \mathrm{d}\mu \le \liminf_{k\to\infty} \int_X g_{m_{l_k}} \mathrm{d}\mu = \liminf_{k\to\infty} \int_X g_k \mathrm{d}\mu.$$

Lehrstuhl für Mathematik IV
Dr Georgios Psaradakis

Universität Mannheim
Fakultät für Wirtschaftsinformatik
und Wirtschaftsmathematik

Tutor: Matthew Liew

Monday 23/11/2020

Advanced analysis

Homework assignment #08

Let μ be a measure on $X \neq \emptyset$. For a given $p \in [1,\infty]$, we write below L^p for $L^p(X,\mu)$.

Exercise 1: Let $1 \leq p < \infty$ and suppose that $f_k \to f$ in L^p. Assume $\{g_k \in L^\infty\}_{k\in\mathbb{N}}$ is such that

(i) $g_k \to g$ as $k \to \infty$, μ-a.e. in X,

(ii) $\sup_{k\in\mathbb{N}} \|g_k\|_\infty \leq M$.

Prove that $f_k g_k \to fg$ in L^p. [2pt]

Exercise 2: Let $1 \leq p \leq \infty$. Prove that if $f_k \to f$ in L^p, then $\|f_k\|_p \to \|f\|_p$ as $k \to \infty$. [2pt]

Exercise 3: Let $1 \leq p < \infty$. Suppose $\{f_k \in L^p\}_{k\in\mathbb{N}}$ and $f \in L^p$ satisfy $f_k \to f$ as $k \to \infty$, μ-a.e. in X, and $\|f_k\|_p \to \|f\|_p$ as $k \to \infty$. Prove $f_k \to f$ in L^p. [2pt]

Exercise 4: Let $1 \leq p \leq \infty$ and suppose $f_k \to f$ in L^p. Prove $f_k \rightharpoonup f$ in L^p. [2pt]

Exercise 5: Suppose $f_k \rightharpoonup f$ in L^2 and $\|f_k\|_{L^2} \to \|f\|_{L^2}$ as $k \to \infty$. Prove $f_k \to f$ in L^2. [2pt]

Upload the solutions on ILIAS by 30/11/20 before 12:00.

Lehrstuhl für Mathematik IV
Dr Georgios Psaradakis

Universität Mannheim
Fakultät für Wirtschaftsinformatik
und Wirtschaftsmathematik

Tutor: Matthew Liew

Monday 30/11/2020

Advanced analysis

Homework assignment #09

For a given $n \in \mathbb{N}$ and $p \in [1,\infty]$, we write below L^p for $L^p(\mathbb{R}^n, \mathscr{L}^n)$.

YOUNG'S INEQUALITY (WITHOUT THE SHARP CONSTANT; SEE LL-§4.2):

Recall that the *convolution* of two $\mathscr{L}^n$-measurable functions f, g is defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\, \mathrm{d}\mathscr{L}^n(y),$$

for any $x \in \mathbb{R}^n$ such that the integral exists.

Exercise 1: Let $p \in [1,\infty]$. Use Fubini's theorem and Hölder's inequality to prove the following simpler version of Young's inequality:

$$f \in L^1,\ g \in L^p \Longrightarrow f * g \in L^p(\mathbb{R}^n, \mathscr{L}^n) \text{ with } \|f * g\|_p \le \|f\|_1 \|g\|_p. \qquad [3pt]$$

Exercise 2: More generally, let $p, q, r \ge 1$ satisfy $1/p + 1/q + 1/r = 2$. Prove *Young's inequality*:

$$f \in L^p,\ g \in L^q,\ h \in L^r \Longrightarrow \textstyle\int_{\mathbb{R}^n} f(x)\big(g * h\big)(x) \mathrm{d}\mathscr{L}^n(x) \le \|f\|_p \|g\|_q \|h\|_r. \qquad [3pt]$$

THE FOURIER TRANSFORM IN L^1:

Recall that for $f \in L^1(\mathbb{R}^n)$, the *Fourier transform* $\hat{f}$ *of* f is defined by

$$\hat{f}(k) := \int_{\mathbb{R}^n} e^{-2\pi i k \cdot x} f(x)\, \mathrm{d}x, \quad k \in \mathbb{R}^n.$$

Exercise 3: Show that the Fourier transform of $f(x) = \chi_{(\alpha,\beta)}(x)$, $x \in \mathbb{R}$, is given by

$$\hat{f}(k) = e^{-\pi i(\alpha+\beta)k} \frac{\sin(\pi(\beta-\alpha)k)}{\pi k}, \quad k \in \mathbb{R},$$

and that this is not an $L^1(\mathbb{R})$ function. [4pt]

Upload the solutions on ILIAS by 07/12/20 before 12:00.