Lehrstuhl für Mathematik IV
HWS 2020/21
Dr. Georgios Psaradakis

Universität Mannheim
Fakultät für Wirtschaftsinformatik
und Wirtschaftsmathematik

Advanced analysis (MAA 508)

Course calendar starting November 09 2020

Monday, 09/11/2020 (10:15-11:45)

We proved Lemma 2.3.10 and explained its relation to Remark 6.0.2, both from our online lecture notes on measure theory.

L^p-SPACES

Let μ be a measure on a set $X \neq \emptyset$.

Notation. From now on we write $\{g > \alpha\}$ for $\{x \in X \mid g(x) > \alpha\}$, etc.

Definition - essential supremum. For a μ-measurable $g : X \to \bar{\mathbb{R}}$ we set

$$\text{ess sup}_X\, g := \begin{cases} 0 & \text{if } \mu(X) = 0, \\ \infty & \text{if } \mu\big(\{g > \alpha\}\big) > 0 \ \ \forall \alpha \in \mathbb{R}, \\ \inf\{\alpha \in \mathbb{R} \mid \mu\big(\{g > \alpha\}\big) = 0\} & \text{otherwise.} \end{cases}$$

Remark. Suppose $\mu\big(\{g > \alpha\}\big) > 0$ for all $\alpha \in \mathbb{R}$. Then $\nexists\ \alpha \in \mathbb{R}$ such that $\mu\big(\{g > \alpha\}\big) = 0$; that is, $\nexists\ \alpha \in \mathbb{R}$ such that $g(x) \leq \alpha$ for μ-a.e. $x \in X$; in other words, $g = \infty$ on a set of positive μ-measure. This justifies the middle definition of ess sup. On the other hand, if $\exists\ \alpha \in \mathbb{R}$ such that $\mu\big(\{g > \alpha\}\big) = 0$, then $g(x) \leq \alpha$ for μ-a.e. $x \in X$; that is α is an upper bound for g (except possibly on a subset of X of μ-measure 0). Hence it is natural to define ess sup as the least upper bound in this case.

Recall that $g : X \to \bar{\mathbb{R}}$ is called μ-*summable* if it is μ-integrable and $\int_X |g| \mathrm{d}\mu < \infty$.

Definition - L^p spaces.

$$L^\infty \equiv L^\infty(X,\mu) := \{\text{all } \mu\text{-measurable functions } f : X \to \bar{\mathbb{R}} \text{ such that ess sup}|f| \text{ is finite}\}.$$

For $p \in (0,\infty)$:

$$L^p \equiv L^p(X,\mu) := \{\text{all } \mu\text{-measurable functions } f : X \to \bar{\mathbb{R}} \text{ such that } |f|^p \text{ is } \mu\text{-summable}\}.$$

Theorem (proved). *Suppose $\mu(X) < \infty$. Then*

(i) $0 < p_1 < p_2 \leq \infty$ *implies* $L^{p_2} \subset L^{p_1}$,

(ii) $\lim_{p\to\infty} \left(\int_X |f|^p \, \mathrm{d}\mu \right)^{1/p} = \text{ess sup}_X |f|$.

Proof. If $f \in L^{p_2}$ and $p_2 < \infty$ then write

$$\begin{aligned} \int_X |f|^{p_1} \, \mathrm{d}\mu &= \int_{\{|f|\leq 1\}} |f|^{p_1} \, \mathrm{d}\mu + \int_{\{|f|>1\}} |f|^{p_1} \, \mathrm{d}\mu \\ &\leq \mu(\{|f| \leq 1\}) + \int_{\{|f|>1\}} |f|^{p_2} \, \mathrm{d}\mu \leq \mu(X) + \int_X |f|^{p_2} \, \mathrm{d}\mu < \infty. \end{aligned}$$

If $f \in L^{p_2}$ and $p_2 = \infty$ then write

$$\int_X |f|^{p_1} \, \mathrm{d}\mu \le \big(\text{ess sup}_X |f|\big)^{p_1} \mu(X) < \infty.$$

this proves (i). For (ii), let $\mu(X) > 0$ and assume first that $\exists\, \alpha \in [0,\infty)$ such that $\mu\big(\{|f| > \alpha\}\big) = 0$. This implies (see the remark following the definition of ess sup) that

$$\text{ess sup}_X |f| = \inf\{\alpha \in \mathbb{R} \mid \mu\big(\{|f| > \alpha\}\big) = 0\} =: M \in [0,\infty).$$

If $\alpha = 0$ or $M = 0$ then $f = 0$ μ-a.e. in X and the result true. If $M > 0$, observe that

$$\Big(\int_X |f|^p \, \mathrm{d}\mu\Big)^{1/p} \le M\big(\mu(X)\big)^{1/p} \Rightarrow \limsup_{p\to\infty} \Big(\int_X |f|^p \, \mathrm{d}\mu\Big)^{1/p} \le M. \;\; (I)$$

On the other hand, given $M' < M$ we have $\mu\big(\{|f| > M'\}\big) > 0$, hence

$$\Big(\int_X |f|^p \, \mathrm{d}\mu\Big)^{1/p} \ge \Big(\int_{\{|f|>M'\}} |f|^p \, \mathrm{d}\mu\Big)^{1/p} \ge M'\big(\mu(\{|f| > M'\})\big)^{1/p} \Rightarrow$$

$$\liminf_{p\to\infty} \Big(\int_X |f|^p \, \mathrm{d}\mu\Big)^{1/p} \ge M'.$$

But this holds true for any $M' < M$, hence

$$\liminf_{p\to\infty} \Big(\int_X |f|^p \, \mathrm{d}\mu\Big)^{1/p} \ge M. \;\; (II)$$

Inequalities (I) and (II) readily imply the result. In the case where $\mu\big(\{|f| > \alpha\}\big) > 0$ for all $\alpha \in [0,\infty)$, then by definition $\text{ess sup}|f| = \infty$. By the remark following the definition of ess sup we know $|f| = \infty$ on a set of positive measure, hence $\int_X |f|^p \mathrm{d}\mu = \infty$ and the result follows. Finally, the case where $\mu(X) = 0$ is also clear since $\text{ess sup}|f| = 0$ by its definition, and also $\int_X |f|^p \mathrm{d}\mu = 0$. ■

Theorem - Hölder's inequality (proved). *Let $p, q \in [1,\infty]$ satisfying $1/p + 1/q = 1$. If $f \in L^p$, $g \in L^q$ then*

$$\int_X |fg| \, \mathrm{d}\mu \le \begin{cases} \big(\text{ess sup}_X |g|\big) \int_X |f| \, \mathrm{d}\mu & \text{if } p = 1, \\ \Big(\int_X |f|^p \, \mathrm{d}\mu\Big)^{1/p} \Big(\int_X |g|^q \, \mathrm{d}\mu\Big)^{1/q} & \text{if } 1 < p < \infty, \\ \big(\text{ess sup}_X |f|\big) \int_X |g| \, \mathrm{d}\mu & \text{if } p = \infty. \end{cases}$$

Proof. Let $1 < p < \infty$ (the cases $p = 1$ and $p = \infty$ are trivial). We know the convexity inequality $|ab| \le \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$ for all $a, b \in \mathbb{R}$ (prove it!). Hence if $\int_X |f|^p \, \mathrm{d}\mu = \int_X |g|^q \, \mathrm{d}\mu = 1$ we get

$$\int_X |fg| \mathrm{d}\mu \le \frac{1}{p}\int_X |f|^p \, \mathrm{d}\mu + \frac{1}{q}\int_X |g|^q \, \mathrm{d}\mu = \frac{1}{p} + \frac{1}{q} = 1 = \Big(\int_X |f|^p \, \mathrm{d}\mu\Big)^{1/p} \Big(\int_X |g|^q \, \mathrm{d}\mu\Big)^{1/q}.$$

If $\int_X |f|^p \, \mathrm{d}\mu,\ \int_X |g|^q \, \mathrm{d}\mu > 0$, normalize f, g as follows

$$\tilde{f} := \frac{f}{\big(\int_X |f|^p \, \mathrm{d}\mu\big)^{1/p}}, \quad \tilde{g} := \frac{g}{\big(\int_X |g|^q \, \mathrm{d}\mu\big)^{1/q}}.$$

Then $\int_X |\tilde{f}|^p \, \mathrm{d}\mu = \int_X |\tilde{g}|^q \, \mathrm{d}\mu = 1$ and as before we have $\int_X |\tilde{f}\tilde{g}| \mathrm{d}\mu \le 1$. Substituting $\tilde{f}$ and $\tilde{g}$ gives the result. ■

Theorem - Minkowski's inequality (proved). *If $f, g \in L^p$ with $p \in [1, \infty)$, then*

$$\left(\int_X |f+g|^p \, \mathrm{d}\mu\right)^{1/p} \leq \left(\int_X |f|^p \, \mathrm{d}\mu\right)^{1/p} + \left(\int_X |g|^p \, \mathrm{d}\mu\right)^{1/p}.$$

If $f, g \in L^\infty$ then $\operatorname{ess\,sup}_X |f+g| \leq \operatorname{ess\,sup}_X |f| + \operatorname{ess\,sup}_X |g|$.

Proof. For $1 < p < \infty$ use the triangle inequality to get

$$\int_X |f+g|^p \, \mathrm{d}\mu \leq \int_X |f||f+g|^{p-1} \, \mathrm{d}\mu + \int_X |g||f+g|^{p-1} \, \mathrm{d}\mu.$$

The proof follows by applying Hölder's inequality with exponents p and $p/(p-1)$ on both terms of the right hand side and then rearranging terms in the resulting inequality. ■

Hence, if $p \in [1, \infty]$, the function $\|\cdot\|_p : L^p \to [0, \infty]$ given by

$$\|f\|_p \equiv \|f\|_{L^p} \equiv \|f\|_{L^p(X,\mu)} := \begin{cases} \left(\int_X |f|^p \, \mathrm{d}\mu\right)^{1/p} & \text{if } p \in [1, \infty) \\ \operatorname{ess\,sup}_X |f| & \text{if } p = \infty \end{cases}, \quad f \in L^p,$$

defines a norm on the linear space L^p.

Friday, 13/11/2020 (12:00-13:30)

Theorem - L^p is a Banach space (proved) - LL§2.7. *Let $1 \leq p \leq \infty$ and suppose $f_k : X \to \bar{\mathbb{R}}$, $k \in \mathbb{N}$, is a Cauchy sequence in L^p. There exists then a subsequence $\{f_{l_k}\}_{k \in \mathbb{N}}$ such that*

(i) *$|f_{l_k}| \leq F$ for all $k \in \mathbb{N}$, μ-a.e. in X, and some nonnegative $F \in L^p$,*

(ii) *$f_{l_k} \to f$ as $k \to \infty$, μ-a.e. in X, and some $f : X \to \bar{\mathbb{R}}$.*

In particular, applying Fatou's lemma to the sequence $g_k := |f_{l_k}|^p$ gives $f \in L^p$. Applying then the dominated convergence theorem for the sequence $h_k := |f_{l_k} - f|^p$, we deduce $\|f_{l_k} - f\|_p \to 0$, as $k \to \infty$. This, together with the fact that $\{f_k\}_{k \in \mathbb{N}}$ is Cauchy in L^p, imply $\|f_k - f\|_p \to 0$, as $k \to \infty$.

THE DUAL OF L^p

Lemma (not proved). *Assume $p \in (1, \infty)$ and for $s \in (0, 1]$, $t \geq 0$ set*

$$\phi(s,t) := h(s) + k(s)t^p, \quad \textit{where} \begin{cases} h(s) := (1+s)^{p-1} + (1-s)^{p-1} \\ k(s) := (1/s+1)^{p-1} - (1/s-1)^{p-1}. \end{cases}$$

Then for any $t \geq 0$ we have

$$\phi(s,t) \leq (1+t)^p + |1-t|^p \quad \textit{for } 1 < p < 2$$

$$\phi(s,t) \geq (1+t)^p + |1-t|^p \quad \textit{for } p > 2.$$

With this at hand we proved

Theorem - Hanner's inequalities (proved) - LL§2.5. *If $f, g \in L^p$ with $p \in [1, 2)$, then*

$$\|f+g\|_p^p + \|f-g\|_p^p \geq \big(\|f\|_p + \|g\|_p\big)^p + \big|\|f\|_p - \|g\|_p\big|^p.$$

$$\big(\|f+g\|_p + \|f-g\|_p\big)^p + \big|\|f+g\|_p - \|f-g\|_p\big|^p \leq 2^p\big(\|f\|_p^p + \|g\|_p^p\big).$$

If $p = 2$ we have equality throughout, and if $p > 2$ the inequalities are reversed.

Proof. Fix $f, g \in L^p$, $1 < p < 2$, and w.l.o.g. assume that $\|f\|_p \geq \|g\|_p > 0$. Take $t = |g|/|f|$, $f \neq 0$, in the above lemma. Multiplying with $|f|^p$ we deduce

$$h(s)|f|^p + k(s)|g|^p \leq \big(|f| + |g|\big)^p + \big||f| - |g|\big|^p = |f+g|^p + |f-g|^p \quad \Rightarrow$$

$$\|f+g\|_p^p + \|f-g\|_p^p \geq h(s)\|f\|_p^p + k(s)\|g\|_p^p.$$

Now compute the right hand side with $s = \|g\|_p/\|f\|_p$ to get

$$\begin{aligned}\|f+g\|_p^p + \|f-g\|_p^p &\geq \big(\|f\|_p + \|g\|_p\big)^p + \big(\|f\|_p + \|g\|_p\big)\big(\|f\|_p - \|g\|_p\big)^{p-1} \\ &\geq \big(\|f\|_p + \|g\|_p\big)^p + \big(\|f\|_p - \|g\|_p\big)^p.\end{aligned}$$

The second inequality follows from the first one by replacing f by $f+g$ and g by $f-g$. The proofs of the counterpart inequalities when $p > 2$ follow the same steps starting from the second inequality of the previous lemma. ■

Lemma (proved). *Let $p \geq 1$. Then for all $a, b \in \mathbb{R}$ there holds $\big(|a| + |b|\big)^p \leq 2^{p-1}\big(|a|^p + |b|^p\big)$.*

Proof. We know $f(t) = t^p$, $t \geq 0$, is convex. In particular, $f\big(|a|/2 + |b|/2\big) \leq 1/2\big(f(|a|) + f(|b|)\big)$ which gives the result. ■

Monday, 16/11/2020 (10:15-11:45)

Theorem - Derivative of the norm (proved) - LL§2.6. *If $f, g \in L^p$ with $p \in (1, \infty)$, then the map $N_{[f,g]} : \mathbb{R} \to [0, \infty)$ given by $N_{[f,g]}(t) := \|f + tg\|_p^p$ satisfies*

$$N'_{[f,g]}(0) = p \int_X |f|^{p-2} f g \, \mathrm{d}\mu.$$

Remark on the proof (exercise 21 in LL). In the proof we used the difference quotient function

$$q(t) := \frac{|f+tg|^p - |f|^p}{t}, \quad 0 < |t| \leq 1,$$

and in particular its property that: $q(-1) \leq q(t) \leq q(1)$ for all $0 < |t| \leq 1$. Here is the proof of this fact: First, using the above lemma we get

$$|2f|^p = |f - tg + f + tg|^p \leq 2^{p-1}\big(|f - tg|^p + |f + tg|^p\big) \quad \Rightarrow$$

$$2|f|^p \leq |f - tg|^p + |f + tg|^p. \tag{1}$$

From the convexity of $t \mapsto |t|^p$ we have $|f - tg|^p = |(1-t)f + t(f-g)|^p \leq (1-t)|f|^p + t|f-g|^p$. Inserting this in (1) readily gives $q(-1) \leq q(t)$ for all $t \in (0,1]$. On the other hand, the convexity of $t \mapsto |t|^p$ also gives $|f + tg|^p = |(1-t)f + t(f+g)|^p \leq (1-t)|f|^p + t|f+g|^p$ which says $q(t) \leq q(1)$ for all $t \in (0,1]$. We have showed $q(-1) \leq q(t) \leq q(1)$ for all $t \in (0,1]$. To get the same estimate for $t \in [-1, 0)$, apply this with $-t$ in place of t and $-g$ in place of g. ■

Lemma - Projection on convex sets (proved) - LL§2.8. *Let K be a closed convex subset of L^p, where $p \in (1, \infty)$. For any $f \in L^p$ there exists $h \in K$ such that*

$$\inf_{g \in K} \|f - g\|_p = \|f - h\|_p.$$

Moreover, there holds $N'_{[f-h,h-g]}(0) \geq 0$ for all $g \in K$.

Friday, 20/11/2020 (12:00-13:30)

In what follows, unless otherwise stated, we assume that $p \in [1, \infty]$.

Definition - Continuous linear functionals of L^p. A *linear functional of* L^p is a map $\ell : L^p \to \mathbb{R}$ for which

$$\ell(\alpha f + \beta g) = \alpha \ell(f) + \beta \ell(g) \quad \forall\, f, g \in L^p, \ \forall\, \alpha, \beta \in \mathbb{R}.$$

Such a functional ℓ is called *continuous* if

$$\lim_{k \to \infty} \ell(f_k) \to 0 \quad \text{whenever } f_k \to 0 \text{ in } L^p,$$

and *bounded* if

$$|\ell(f)| \leq K \|f\|_p \quad \forall\, f \in L^p.$$

Proposition (proved). *A linear functional ℓ of L^p is continuous if and only if*

(i) $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ *such that* $|\ell(f)| \leq \varepsilon$ *whenever* $\|f\|_p \leq \delta$,

(ii) *it is bounded.*

Proof. (i) Let ℓ be a continuous linear functional of L^p and suppose in contrary that there exists $\varepsilon > 0$ such that:

$$\forall\, \delta > 0, \ \exists\, f_\delta \in L^p \text{ with } \|f_\delta\|_p \leq \delta \text{ but } |\ell(f_\delta)| > \varepsilon.$$

Taking $\delta = 1/k$, $k \in \mathbb{N}$, we obtain a sequence $\{f_k\}_{k \in \mathbb{N}}$ such that $\|f_k\|_p \leq 1/k$ but $|\ell(f_k)| > \varepsilon$. Letting $k \to \infty$ we see that $f_k \to 0$ in L^p but $|\ell(f_k)| > \varepsilon$ for all $k \in \mathbb{N}$, a contradiction to the continuity of ℓ. For the reverse statement, let $\varepsilon > 0$ and suppose ℓ is a linear functional of L^p such that

$$\exists\, \delta(\varepsilon) > 0 \text{ such that } |\ell(f)| \leq \varepsilon \text{ whenever } \|f\|_p \leq \delta(\varepsilon). \tag{1}$$

For a sequence such that $f_k \to 0$ in L^p, we know

$$\exists\, k(\delta(\varepsilon)) \in \mathbb{N} \text{ such that } \|f_k\|_p \leq \delta(\varepsilon) \ \forall\, k \geq k(\delta(\varepsilon)) \implies^{(1)}$$

$$\exists\, k(\delta(\varepsilon)) \in \mathbb{N} \text{ such that } |\ell(f_k)| \leq \varepsilon \ \forall\, k \geq k(\delta(\varepsilon)).$$

Hence, given $\varepsilon > 0$ we have found $\tilde{k}(\varepsilon) := k(\delta(\varepsilon)) \in \mathbb{N}$ such that $|\ell(f_k)| \leq \varepsilon$ for all $k \geq \widetilde{k(\varepsilon)}$; this means $\ell(f_k) \to 0$.

(ii) Let ℓ be a continuous linear functional of L^p. From (ii) with $\varepsilon = 1$, we get

$$\exists\, \delta > 0 \text{ such that } |\ell(f)| \leq 1 \text{ whenever } \|f\|_p \leq \delta.$$

But for any $f \in L^p \setminus \{0\}$ we have that $\tilde{f} := \delta f / \|f\|_p$ satisfies $\|\tilde{f}\|_p = \delta$. Hence $|\ell(\tilde{f})| \leq 1$; this gives $|\ell(f)| \leq (1/\delta)\|f\|_p$. Since this estimate is true also when $f = 0$, we conclude that $|\ell(f)| \leq K\|f\|_p$ for all $f \in L^p$, where $K = 1/\delta$; that is, ℓ is bounded. For the reverse statement, if ℓ is a bounded linear functional of L^p then given any $\{f_k \in L^p\}_{k \in \mathbb{N}}$ we know $|\ell(f_k)| \leq K\|f_k\|_p$ for all $k \in \mathbb{N}$. In particular, if $f_k \to 0$ in L^p, then $|\ell(f_k)| \to 0$ as $k \to \infty$; hence ℓ is continuous. ■

Definition - Dual space. The set of all bounded linear functionals of L^p is called the *dual* of L^p and is denoted by $\left(L^p\right)^*$. It is a normed linear space with the norm

$$\|\ell\| = \sup\{|\ell(f)| \ \big| \ \|f\|_p \leq 1\}.$$

Theorem - The dual of L^p - LL§2.14.

(i) ***(proved)*** *If $p \in (1,\infty)$ then $(L^p)^* = L^{p/(p-1)}$ in the sense that*

$$\forall\, \ell \in (L^p)^*,\ \ \exists!\ v_\ell \in L^{p/(p-1)} \text{ such that } \ell(g) = \int_X v_\ell g \,\mathrm{d}\mu \quad \forall\, g \in L^p.$$

(ii) *If $p = 1$ then $(L^1)^* = L^\infty$ in the above sense provided that X is σ-finite with respect to μ.*

(iii) ***(proved)*** *If $p \in [1,\infty]$, a functional $\ell : L^p \to \mathbb{R}$ defined for all $g \in L^p$ by*

$$\ell(g) = \int_X vg \,\mathrm{d}\mu \quad \text{for some } v \in L^q \text{ where } 1/p + 1/q = 1, \tag{2}$$

is always a member of $(L^p)^*$ and moreover $\|\ell\| = \|v\|_q$.

Remark 0.1. *Assertions* (i) *and* (ii) *say that when $p \in [1,\infty)$ then all bounded linear functionals you can find are of the form* (2) *(provided some σ-finiteness condition on X when $p = 1$). From assertion* (iii) *we understand in particular that when $p = \infty$, functionals of the form* (2) *with $v \in L^1$ are members of the dual space $(L^\infty)^*$, but these are not all one can find; that is $(L^\infty)^* \supsetneq L^1$.*[1]

Monday, 23/11/20 (10:15-11:45)

We proved part (ii) of the above theorem.

WEAK CONVERGENCE IN L^p:

Definition - Weak convergence in L^p - LL§2.9. Let $1 \le p \le \infty$. A sequence $\{f_k \in L^p\}_{k\in\mathbb{N}}$ is said to *converge weakly in L^p* to $f \in L^p$, denoted by $f_k \rightharpoonup f$ in L^p, whenever

$$\ell(f_k) \to \ell(f) \qquad \text{for all } \ell \in (L^p)^*.$$

Observe that for $1 < p < \infty$, by the "dual of L^p" Theorem we get that $f_k \rightharpoonup f$ in L^p means exactly that

$$\lim_{k\to\infty} \int_X (f_k - f) g \,\mathrm{d}\mu = 0 \qquad \text{for all } g \in L^q, \text{ where } q \text{ is given by } 1/p + 1/q = 1,$$

and that the same is true for $p = 1$ provided that X is σ-finite with respect to μ.

Proposition (proved) - LL§2.10. *Let $f \in L^p$, $p \in [1,\infty)$. Then*

$$l(f) = 0 \quad \forall\, l \in (L^p)^* \Longrightarrow \quad f = 0 \ \ \mu\text{-a.e. in } X.$$

This holds true also in case $p = \infty$ provided X is σ-finite.

Corollary (proved) - LL§2.10. *Let $p \in [1,\infty)$ and $f_k \rightharpoonup g$ in L^p, $f_k \rightharpoonup h$ in L^p. Then $g = h$ μ-a.e. in X. This holds true also in case $p = \infty$ provided X is σ-finite.*

Friday, 27/11/2020 (12:00-13:30)

Theorem - Lower semi-continuity of norms (proved) - LL§2.11. *Suppose that $f_k \rightharpoonup g$ in L^p, $p \in [1,\infty]$. Then*

[1] For the exact characterization of $(L^\infty)^*$ we refer to ***Dunford, N.; Schwartz, J. T. - Linear Operators. Part I: General Theory. (pg 296, 15 Definition & 16 Theorem)*** and ***Yosida, K. - Functional Analysis. 6th edition. (pg 118, Example 5)***.

(i) *if* $p \in [1,\infty)$, *then* $\liminf_{k\to\infty} \|f_k\|_p \geq \|f\|_p$,

(ii) *if* $p = \infty$ *the same is true provided* X *is* σ*-finite,*

(iii) *if* $p \in (1,\infty)$ *and* $\lim_{k\to\infty} \|f_k\|_p = \|f\|_p$, *then* $f_k \to f$ *in* L^p.

Theorem - Uniform boundedness principle (proved) - LL§2.12. *Let* $p \in [1,\infty]$ *and suppose* $\{f_k \in L^p\}_{k\in\mathbb{N}}$ *is such that* $\{l(f_k)\}_{k\in\mathbb{N}}$ *is bounded for any* $l \in (L^p)^*$. *Then* $\{\|f_k\|_p\}_{k\in\mathbb{N}}$ *is also bounded.*

Monday, 30/11/2020 (10:15-11:45)

Suppose that $f_k \rightharpoonup f$ in L^p, where $p \in [1,\infty]$; that is $\lim_{k\to\infty} \ell(f_k) = \ell(f)$ for all $\ell \in (L^p)^*$ and so $\{l(f_k)\}_{k\in\mathbb{N}}$ is bounded for any $l \in (L^p)^*$. It follows from the uniform boundedness principle that $\{\|f_k\|_p\}_{k\in\mathbb{N}}$ is bounded. Thus,

$$f_k \rightharpoonup f \text{ in } L^p \Longrightarrow \{\|f_k\|_p\}_{k\in\mathbb{N}} \text{ is bounded.}$$

Our next target is to establish the converse of this fact, modulo passing to a subsequence, in the special case where $\mu = \mathscr{L}^n$, $X \subseteq \mathbb{R}^n$ is $\mathscr{L}^n$-measurable and $p \in (1,\infty]$. To do so we need the following important assertion:

Theorem[2] - Separability of L^p - LL§2.17. *Let* $X \subseteq \mathbb{R}^n$ *be* $\mathscr{L}^n$*-measurable. Then* $L^p(X,\mathscr{L}^n)$ *for* $1 \leq p < \infty$ *is separable, i.e., it contains a countable dense subset (this fails for* $p = \infty$*).*

Accepting this, we proved first

Theorem - Weak compactness in L^p/Banach-Alaoglu theorem (proved) - LL§2.18. *Let* $X \subseteq \mathbb{R}^n$ *be* $\mathscr{L}^n$*-measurable and* $p \in (1,\infty)$. *If* $\{f_k \in L^p(X,\mathscr{L}^n)\}_{k\in\mathbb{N}}$ *is such that for some* $M > 0$ *we have* $\|f_k\|_p \leq M$ *for all* $k \in \mathbb{N}$, *then there exists* $f \in L^p(X,\mathscr{L}^n)$ *and a subsequence* $\{f_{l_k}\}_{k\in\mathbb{N}}$ *such that* $f_{l_k} \rightharpoonup f$ *in* $L^p(X,\mathscr{L}^n)$.

Proof. We assume initially that $p \in (1,\infty]$. It is only the last argument of the proof that does not work for $p = \infty$. Since $p \in (1,\infty]$ implies $q \in [1,\infty)$, we get from the "separability of L^p"-theorem that $L^q(X,\mathscr{L}^n) = \overline{S}$ where $S := \{g_j\}_{k\in\mathbb{N}} \subset L^q(X,\mathscr{L}^n)$. Now for each $j \in \mathbb{N}$ consider the sequence of numbers

$$F_k(g_j) := \int_X f_k g_j \, \mathrm{d}\mathscr{L}^n, \qquad k \in \mathbb{N}. \qquad (1)$$

Holder's inequality implies

$$|F_k(g_j)| \leq \|f_k\|_p \|g_j\|_q \leq M\|g_j\|_q \quad \forall\, k \in \mathbb{N}. \qquad (2)$$

Start with $j = 1$ and proceed as follows:

- (2) says that the sequence of numbers $\{F_k(g_1)\}_{k\in\mathbb{N}}$ is bounded. It contains thus a convergent subsequence, denoted by $\{F_k^{(1)}(g_1)\}_{k\in\mathbb{N}}$. Because of (1), we have in fact obtained a subsequence $\{f_k^{(1)}\}_{k\in\mathbb{N}}$ such that the sequence of numbers $\{F_k^{(1)}(g_1) = \int_X f_k^{(1)} g_1 \, \mathrm{d}\mathscr{L}^n\}_{k\in\mathbb{N}}$ converges, say to $l_1 \in \mathbb{R}$.
- (2) says that the sequence of numbers $\{F_k^{(1)}(g_2)\}_{k\in\mathbb{N}}$ is bounded. It contains thus a convergent subsequence, denoted by $\{F_k^{(2)}(g_2)\}_{k\in\mathbb{N}}$. Because of (1), we have in fact obtained a subsequence

[2] We proved this after the Banach-Alaoglu theorem and we had time only for the first part of its proof; i.e. $L^p(X,\mathscr{L}^n) \cap L^\infty(X,\mathscr{L}^n)$ when $\mathscr{L}^n(X) < \infty$ is separable.

$\{f_k^{(2)}\}_{k\in\mathbb{N}}$ such that the sequence of numbers $\{F_k^{(2)}(g_2) = \int_X f_k^{(2)} g_2 \, \mathrm{d}\mathscr{L}^n\}_{k\in\mathbb{N}}$ converges, say to $l_2 \in \mathbb{R}$.

- Note that $\{F_k^{(2)}(g_1)\}_{k\in\mathbb{N}}$ is a subsequence of $\{F_k^{(1)}(g_1)\}_{k\in\mathbb{N}}$ and hence converges to l_1 too.

This way we arrive to a sequence of subsequences $\big\{\{f_k^{(m)}\}_{k\in\mathbb{N}}\big\}_{m\in\mathbb{N}}$ such that for any $m \in \mathbb{N}$: (i) $\{f_k^{(m+1)}\}_{k\in\mathbb{N}}$ is a subsequence of $\{f_k^{(m)}\}_{k\in\mathbb{N}}$ and (ii) $\{F_k^{(m)}(g_j) = \int_X f_k^{(m)} g_j \, \mathrm{d}\mathscr{L}^n\}_{k\in\mathbb{N}}$ converges for all $j = 1,\dots,m$. In particular, if $l_m := \lim_{k\to\infty} F_k^{(m)}(g_m)$, then $\lim_{k\to\infty} F_k^{(m)}(g_j) = l_j$ for all $j = 1,\dots,m-1$. For instance, we get for $j = 1$:

$$\begin{array}{llllll}
F_1^{(1)}(g_1) & F_2^{(1)}(g_1) & F_3^{(1)}(g_1) & \dots & & \text{converges to } l_1 \\
F_1^{(2)}(g_1) & F_2^{(2)}(g_1) & F_3^{(2)}(g_1) & \dots & & \text{converges to } l_1 \\
F_1^{(3)}(g_1) & F_2^{(3)}(g_1) & F_3^{(3)}(g_1) & \dots & & \text{converges to } l_1 \\
\dots & \dots & \dots & & &
\end{array}$$

Clearly, the diagonal sequence $\{G_k(g_1) := F_k^{(k)}(g_1)\}_{k\in\mathbb{N}}$ converges to l_1 too. In the same fashion we construct convergent sequences $\{G_k(g_j) := F_k^{(k)}(g_j) = \int_X f_k^{(k)} g_j \, \mathrm{d}\mathscr{L}^n\}_{k\in\mathbb{N}}$ for all $j \in \mathbb{N}$.

Now, for a given $g \in L^q(X,\mathscr{L}^n)$ set $G_k(g) := \int_X f_k^{(k)} g \, \mathrm{d}\mathscr{L}^n$. Fix $\varepsilon > 0$ and let $g_j \in S$ be such that $\|g - g_j\|_p < \varepsilon$. Since $\{G_k(g_j)\}_{k\in\mathbb{N}}$ converges, it is Cauchy; that is

$$|G_k(g_j) - G_m(g_j)| < \varepsilon \qquad \forall\, k,m \geq N_\varepsilon.$$

Therefore, using (2) twice,

$$\begin{aligned}
|G_k(g) - G_m(g)| &\leq |G_k(g - g_j)| + |G_m(g - g_j)| + |G_k(g_j) - G_m(g_j)| \\
&\leq M\|g - g_j\|_p + M\|g - g_j\|_p + \varepsilon \\
&\leq (2M+1)\varepsilon \qquad\qquad\qquad\qquad \forall\, k,m \geq N_\varepsilon.
\end{aligned}$$

It follows that for any $g \in L^q(X,\mathscr{L}^n)$ the sequence $\{G_k(g)\}_{k\in\mathbb{N}}$ is Cauchy and hence convergent. Write $F(g) := \lim_{k\to\infty} G_k(g)$, $g \in L^q(X,\mathscr{L}^n)$. It is elementary to see that F is a bounded linear functional of L^q. By the "dual of L^p"-theorem we obtain $f \in L^p(X,\mathscr{L}^n)$ such that $F(g) = \int_X fg \, \mathrm{d}\mathscr{L}^n$ for all $g \in L^q(X,\mathscr{L}^n)$. Consequently, $\int_X fg \, \mathrm{d}\mathscr{L}^n = \lim_{k\to\infty} \int_X f_k^{(k)} g \, \mathrm{d}\mathscr{L}^n$ for all $g \in L^q(X,\mathscr{L}^n)$; that is, we 've found $f \in L^p(X,\mathscr{L}^n)$ and a subsequence $\{f_{l_k} := f_k^{(k)}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \int_X (f_{l_k} - f) g \, \mathrm{d}\mathscr{L}^n = 0 \qquad \forall\, g \in L^q(X,\mathscr{L}^n). \qquad (\star)$$

If $p \neq \infty$, again by the "dual of L^p"-theorem, $(\star)$ means precisely that $f_{l_k} \rightharpoonup f$ in $L^p(X,\mathscr{L}^n)$. ∎

Thursday, 03/12 (10:15-11:45)

We finished the proof of the "Separability" of L^p theorem.

Corollary (proved) - Density of C_c in L^p. *Let $X \subseteq \mathbb{R}^n$ be $\mathscr{L}^n$-measurable. Then $C_c(X)$ is dense in $L^p(X,\mathscr{L}^n)$ for $1 \leq p < \infty$.*

THE FOURIER TRANSFORM, LL-§5:

Definition - Fourier transform in L^1. For $f \in L^1(\mathbb{R}^n)$, the *Fourier transform $\hat{f}$ of f* is given by

$$\hat{f}(k) := \int_{\mathbb{R}^n} e^{-2\pi i k\cdot x} f(x) \, \mathrm{d}\mathscr{L}^n(x), \quad k \in \mathbb{R}^n.$$

Proposition (proved) - Properties of $\hat{f}$ - LL§5.1 & §5.2.

(i) $\hat{f} \in L^\infty(\mathbb{R}^n)$ *with* $\|\hat{f}\|_\infty \le \|f\|_1$. *In particular, for nonnegative* f *we clearly have* $\hat{f}(0) = \|f\|_1$, *hence* $\|\hat{f}\|_\infty = \|f\|_1$ *in this case.*

(ii) $\widehat{\alpha f + \beta g} = \alpha \hat{f} + \beta \hat{g}$ *for all* $f, g \in L^1(\mathbb{R}^n)$ *and all* $\alpha, \beta \in \mathbb{C}$.

(iii) $\widehat{f * g}(k) = \hat{f}(k)\hat{g}(k)$ *for all* $f, g \in L^1(\mathbb{R}^n)$, *where* $f * g(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}y$.

(iv) $\int_{\mathbb{R}^n} \hat{f} g \,\mathrm{d}\mathscr{L}^n = \int_{\mathbb{R}^n} f\hat{g}\,\mathrm{d}\mathscr{L}^n$ *for all* $f, g \in L^1(\mathbb{R}^n)$.

(v) $\widehat{\tau_h f}(k) = e^{-2\pi i k\cdot h}\hat{f}(k)$ *for all* $h \in \mathbb{R}^n$, *where* $\big(\tau_h f\big)(x) := f(x-h)$.

(vi) $\widehat{\delta_\lambda f}(k) = \lambda^n \hat{f}(\lambda k)$ *for all* $\lambda \in (0,\infty)$, *where* $\big(\delta_\lambda f\big)(x) := f(x/\lambda)$.

(vii) $\widehat{e^{2\pi i y\cdot x} g(y)}(k) = \tau_x\big(\hat{g}(k)\big)$ *for all* $x \in \mathbb{R}^n$ *and all* $g \in L^1(\mathbb{R}^n)$.

(viii) $\hat{f} \in C(\mathbb{R}^n)$.

(ix) $\lim_{|k|\to\infty} \hat{f}(k) = 0$.

(x) $\widehat{e^{-\pi\lambda|x|^2}}(k) = \lambda^{-n/2} e^{-\pi|k|^2/\lambda}$ *for all* $\lambda \in (0,\infty)$.

proof of (x). [3] By direct computation

$$\widehat{e^{-\pi\lambda|x|^2}}(k) = \int_{\mathbb{R}^n} e^{-2\pi i k\cdot x - \pi\lambda|x|^2}\,\mathrm{d}x = e^{-\pi|k|^2/\lambda}\int_{\mathbb{R}^n} e^{-|i\sqrt{\pi/\lambda}k + \sqrt{\pi\lambda}x|^2}\,\mathrm{d}x$$

$$= (\pi\lambda)^{-n/2} e^{-\pi|k|^2/\lambda}\int_{\mathbb{R}^n} e^{-|y|^2}\,\mathrm{d}y = (\pi\lambda)^{-n/2} e^{-\pi|k|^2/\lambda}\Big(\int_{\mathbb{R}} e^{-t^2}\,\mathrm{d}t\Big)^n.$$

Finally notice that $\int_{\mathbb{R}} e^{-t^2}\,\mathrm{d}t = \sqrt{\pi}$. ■

Lemma (proved) - Continuity of the translation operator in L^p. *Let* $p \in [1,\infty)$. *For any* $f \in L^p(\mathbb{R}^n, \mathscr{L}^n)$ *we have* $\lim_{h\to 0}\|\tau_h f - f\|_p = 0$.

Friday, 04/12 (12:00-13:30)

By using the dominated convergence theorem, the continuity of the translation operator in L^1, as well as a couple of the aforementioned properties of the Fourier transform of L^1 functions, we proved the following:

Proposition - Invertibility of the Fourier transform (proved). *If* $f, \hat{f} \in L^1(\mathbb{R}^n)$ *then*

$$f(x) = \int_{\mathbb{R}^n} e^{2\pi i k\cdot x}\hat{f}(k)\,\mathrm{d}k\big(= \hat{\hat{f}}(-x)\big), \quad \textit{for a.e. } x \in \mathbb{R}^n.$$

Note however that $f \in L^1(\mathbb{R}^n) \not\Rightarrow \hat{f} \in L^1(\mathbb{R}^n)$ (consider for instance $f = \chi_{(\alpha,\beta)}$ in $\mathbb{R}$).

Proof. For any $\varepsilon > 0$ and any $x \in \mathbb{R}^n$ we set

$$J_\varepsilon(x) := \int_{\mathbb{R}^n} e^{2\pi i k\cdot x - \varepsilon\pi|k|^2}\hat{f}(k)\,\mathrm{d}\mathscr{L}^n(k).$$

[3] more direct than the one in the book but requires complex integration

Using first property (iv), then property (vii) with $g(y) := e^{-\varepsilon\pi|k|^2}$ and finally property (x) with $\lambda = \varepsilon$, we deduce

$$J_\varepsilon(x) = \varepsilon^{-n/2} \int_{\mathbb{R}^n} e^{-\pi|k-x|^2/\varepsilon} f(k)\, \mathrm{d}\mathscr{L}^n(k).$$

Because of Fubini's theorem and the continuity of the translation operator in L^1, we can see that $J_\varepsilon(x)$ converges to $f(x)$ in $L^1(\mathbb{R}^n)$ (see below, in the proof of "Approximation in L^p theorem", for the details). Hence, for some subsequence $\delta(\varepsilon)$ we know $J_{\delta(\varepsilon)}(x) \to f(x)$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$, as $\varepsilon \to 0$. On the other hand, the integrand of $J_{\delta(\varepsilon)}(x)$ converges to $e^{2\pi i k\cdot x}\hat{f}(k)$ as $\varepsilon \to 0$, for all $x \in \mathbb{R}^n$. Moreover, this same integrand is dominated by $|\hat{f}(k)|$ which is (by assumption) an $L^1(\mathbb{R}^n, \mathrm{d}\mathscr{L}^n(k))$ function. So the dominated convergence theorem applies to end the proof. ∎

Lemma (not proved) - Continuous version of the Minkowski inequality. *Let $p \in [1,\infty)$. For any $f \in L^p(\mathbb{R}^n \times \mathbb{R}^n, \mathscr{L}^n)$ we have*

$$\left(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |f(x,y)|\, \mathrm{d}\mathscr{L}^n(y)\Big)^p \mathrm{d}\mathscr{L}^n(x)\right)^{1/p} \le \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |f(x,y)|^p\, \mathrm{d}\mathscr{L}^n(x)\Big)^{1/p} \mathrm{d}\mathscr{L}^n(y).$$

Proof. Set $F(x) := \int_{\mathbb{R}^n} |f(x,y)|\, \mathrm{d}\mathscr{L}^n(y)$. Then, writing $\big(F(x)\big)^p = F(x)\big(F(x)\big)^{p-1}$, we get

$$\begin{aligned}\mathfrak{I} := \int_{\mathbb{R}^n} \big(F(x)\big)^p\, \mathrm{d}\mathscr{L}^n(x) &= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |f(x,y)|\, \mathrm{d}\mathscr{L}^n(y)\Big)\big(F(x)\big)^{p-1}\, \mathrm{d}\mathscr{L}^n(x)\\ &= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |f(x,y)|\big(F(x)\big)^{p-1}\, \mathrm{d}\mathscr{L}^n(x)\Big)\, \mathrm{d}\mathscr{L}^n(y),\end{aligned}$$

from Fubini's theorem. Applying now Hölder's inequality on the inner integral we deduce

$$\begin{aligned}\mathfrak{I} &\le \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |f(x,y)|^p\, \mathrm{d}\mathscr{L}^n(x)\Big)^{1/p} \Big(\int_{\mathbb{R}^n} \big(F(x)\big)^p\, \mathrm{d}\mathscr{L}^n(x)\Big)^{(p-1)/p} \mathrm{d}\mathscr{L}^n(y)\\ &= \mathfrak{I}^{(p-1)/p} \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |f(x,y)|^p\, \mathrm{d}\mathscr{L}^n(x)\Big)^{1/p} \mathrm{d}\mathscr{L}^n(y).\end{aligned}$$

Hence $\mathfrak{I}^{1/p} \le \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |f(x,y)|^p\, \mathrm{d}\mathscr{L}^n(x)\Big)^{1/p} \mathrm{d}\mathscr{L}^n(y)$ which is the desired inequality. ∎

Using the last two lemmata (continuity of the translation operator in L^p and continuous version of Minkowski's inequality) one has the following useful approximation theorem, already used with $p = 1$ in the "Invertibility of the Fourier transform".

Theorem (proved) - Approximation in L^p. Let $f \in L^p(\mathbb{R}^n)$, $p \in [1,\infty)$. Then,

$$\lim_{\varepsilon \to 0} \left\| \varepsilon^{-n/2} \int_{\mathbb{R}^n} e^{-\pi|k-x|^2/\varepsilon} f(k)\, \mathrm{d}\mathscr{L}^n(k) - f(x) \right\|_{L^p\left(\mathbb{R}^n, \mathrm{d}\mathscr{L}^n(x)\right)} = 0.$$

Proof. Let

$$I_\varepsilon := \left\| \varepsilon^{-n/2} \int_{\mathbb{R}^n} e^{-\pi|k-x|^2/\varepsilon} f(k)\, \mathrm{d}\mathscr{L}^n(k) - f(x) \right\|_{L^p\left(\mathbb{R}^n, \mathrm{d}\mathscr{L}^n(x)\right)}.$$

Since $\varepsilon^{-n/2} \int_{\mathbb{R}^n} e^{-\pi|k-x|^2/\varepsilon}\, \mathrm{d}\mathscr{L}^n(k) = 1$, we have

$$I_\varepsilon = \left\| \pi^{-n/2} \int_{\mathbb{R}^n} e^{-|y|^2} \big(f(x + \sqrt{\varepsilon/\pi}\, y) - f(x)\big)\, \mathrm{d}\mathscr{L}^n(y) \right\|_{L^p\left(\mathbb{R}^n, \mathrm{d}\mathscr{L}^n(x)\right)},$$

where we have also changed variables by $k = x + \sqrt{\varepsilon/\pi}y$. By the continuous version of the Minkowski inequality, or just by the Fubini theorem in case $p = 1$, we arrive at

$$I_\varepsilon \leq \pi^{-n/2} \int_{\mathbb{R}^n} e^{-|y|^2} \Big(\int_{\mathbb{R}^n} \big|f(x+\sqrt{\varepsilon/\pi}y) - f(x)\big|^p \, \mathrm{d}\mathscr{L}^n(x) \Big)^{1/p} \mathrm{d}\mathscr{L}^n(y). \qquad (\star)$$

But $h(\varepsilon) := \sqrt{\varepsilon/\pi}y \to 0$ as $\varepsilon \to 0$. Hence from the continuity of the translation operator in L^p we get that $\lim_{\varepsilon\to 0} \|\tau_{h(\varepsilon)} f - f\|_p = 0$; that is, the integrand on the right term of $(\star)$ vanishes as $\varepsilon \to 0$. Using Minkowski's inequality we easily see that this same integrand is dominated by $2\|f\|_p \, e^{-|y|^2}$ which is an $L^1\big(\mathbb{R}^n, \mathrm{d}\mathscr{L}^n(y)\big)$ function. So the dominated convergence theorem applies to end the proof. ■

We used the above theorem with $p = 2$ in the proof of the following fundamental result:

Theorem (proved) - Plancherel's formula - LL§5.3. *If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $\hat{f} \in L^2(\mathbb{R}^n)$ with $\|\hat{f}\|_2 = \|f\|_2$.*

Monday, 07/12 (10:15-11:45)

Definition - Fourier transform in L^2- LL§5.3 & §5.4 & §5.5. Given $f \in L^2(\mathbb{R}^n)$, think of a sequence $\{f_j \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\}_{j\in\mathbb{N}}$ such that $f_j \to f$ in L^2. By Plancherel's formula we get then $\|\hat{f}_j - \hat{f}_l\|_2 = \|f_j - f_l\|_2$ for all $j, l \in \mathbb{N}$; that is, $\{\hat{f}_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in L^2. But L^2 is complete and thus $\{\hat{f}_j\}_{j\in\mathbb{N}}$ converges to a function of $L^2(\mathbb{R}^n)$ which we call the Fourier transform of f and denote it by $\hat{f}$.

Remark Given $f \in L^2(\mathbb{R}^n)$ we can always find sequences $\{f_j \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\}_{j\in\mathbb{N}}$ such that $f_j \to f$ in L^2. For example, taking $f_j := \eta_{1/j} * f$, where η_ε for $\varepsilon > 0$ is the standard mollifier, we have $\{f_j \in C_c^\infty(\mathbb{R}^n)\}_{j\in\mathbb{N}}$ such that $f_j \to f$ in L^2. Another example is $\{f_j := f\chi_{B_j}\}_{j\in\mathbb{N}}$. Since by Hölder's inequality $\|f_j\|_1 \leq \|f\|_2 [\mathscr{L}^n(B_j)]^{1/2}$ for all $j \in \mathbb{N}$ and also $\|f_j\|_2 \leq \|f\|_2$ for all $j \in \mathbb{N}$, we have $f_j \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ for all $j \in \mathbb{N}$. Moreover,

$$\|f_j - f\|_2^2 = \int_{\mathbb{R}^n} g_j \, \mathrm{d}\mathscr{L}^n, \qquad \text{where } g_j := (1 - \chi_{B_j})|f|^2.$$

Since $\lim_{j\to\infty} g_j \to 0$ $\mathscr{L}^n$-a.e. in $\mathbb{R}^n$ and $g_j \leq |f|^2 \in L^1(\mathbb{R}^n)$, the dominated convergence theorem readily gives $f_j \to f$ in L^2. This last example provides us with a fairly simple sequence of functions whose L^2-limit defines $\hat{f}$:

$$\Big\{ \int_{B_j} e^{-2\pi i k\cdot x} f(x) \, \mathrm{d}\mathscr{L}^n(x) \Big\}_{j\in\mathbb{N}}.$$

Remark Although there are many sequences such that $f_j \to f$ in L^2, $\hat{f}$ is independent of the one we choose. Indeed, suppose $\{f_j \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\}_{j\in\mathbb{N}}$ satisfies $\lim_{j\to\infty} \|f_j - f\|_2 = 0$. Then $\hat{f}$ is defined as the L^2-limit of $\hat{f}_j$, hence $\lim_{j\to\infty} \|\hat{f}_j - \hat{f}\|_2 = 0$. Let $\{g_j \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\}_{j\in\mathbb{N}}$ be one more sequence such that $\lim_{j\to\infty} \|g_j - f\|_2 = 0$. Then

$$\|\hat{g}_j - \hat{f}\|_2 \leq \|\hat{g}_j - \hat{f}_j\|_2 + \|\hat{f}_j - \hat{f}\|_2 = \|g_j - f_j\|_2 + \|\hat{f}_j - \hat{f}\|_2 \leq \|g_j - f\|_2 + \|f_j - f\|_2 + \|\hat{f}_j - \hat{f}\|_2,$$

where we have used Plancherel's formula to get the middle equality. Thus, $\lim_{j\to\infty} \|\hat{g}_j - \hat{f}\|_2 = 0$ which says $\hat{f}$ is the L^2-limit of $\{\hat{g}_j\}_{j\in\mathbb{N}}$ too.

Proposition - Properties of $\hat{f}$.

(i) *(isometry) If $f \in L^2(\mathbb{R}^n)$, then $\|\hat{f}\|_2 = \|f\|_2$.*

Proof: Suppose $\{f_j \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\}_{j\in\mathbb{N}}$ satisfies $\lim_{j\to\infty} \|f_j - f\|_2 = 0$, hence $\|f\|_2 = \lim_{j\to\infty} \|f_j\|_2$. But $\hat{f}$ is defined as the L^2-limit of $\hat{f}_j$; that is, $\lim_{j\to\infty} \|\hat{f}_j - \hat{f}\|_2 = 0$, hence $\|\hat{f}\|_2 = \lim_{j\to\infty} \|\hat{f}_j\|_2$. Now the proof follows by Plancherel's formula: $\|\hat{f}_j\|_2 = \|f_j\|_2$ by letting $j \to \infty$.

(ii) *(linearity) If $f, g \in L^2(\mathbb{R}^n)$ and $\alpha, \beta \in \mathbb{C}$, then $\widehat{\alpha f + \beta g} = \alpha \hat{f} + \beta \hat{g}$.*

(iii) *(Parseval's formula) If $f, g \in L^2(\mathbb{R}^n)$, then $\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$, where*

$$\langle f, g\rangle := \int_{\mathbb{R}^n} \bar{f} g \,\mathrm{d}\mathscr{L}^n.$$

(iii) *(invertibility) If $f \in L^2(\mathbb{R}^n)$, then $f(x) = \hat{\hat{f}}(-x)$.*

Proof: Suppose $\{f_j \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\}_{j\in\mathbb{N}}$ satisfies $\lim_{j\to\infty} \|f_j - f\|_2 = 0$. As in the proof of the Proposition - Invertibility of the Fourier transform, we have

$$\int_{\mathbb{R}^n} e^{2\pi i k\cdot x - \varepsilon\pi|k|^2} \hat{f}_j(k) \,\mathrm{d}\mathscr{L}^n(k) = \varepsilon^{-n/2} \int_{\mathbb{R}^n} e^{-\pi|k-x|^2/\varepsilon} f_j(k) \,\mathrm{d}\mathscr{L}^n(k) \quad \forall\, j \in \mathbb{N}, \forall\, \varepsilon > 0. \quad (\star)$$

By Hölder's inequality,

$$\Big|\varepsilon^{-n/2} \int_{\mathbb{R}^n} e^{-\pi|k-x|^2/\varepsilon} \big(f_j(k) - f(k)\big) \,\mathrm{d}\mathscr{L}^n(k)\Big| \le \varepsilon^{-n/2} \Big(\int_{\mathbb{R}^n} e^{-2\pi|k-x|^2/\varepsilon} \mathrm{d}k\Big)^{1/2} \|f_j - f\|_2 \to 0,$$

as $j \to \infty$. Also $\hat{f}$ is defined as the L^2-limit of $\hat{f}_j$; that is, $\lim_{j\to\infty} \|\hat{f}_j - \hat{f}\|_2 = 0$. Hence,

$$\Big|\int_{\mathbb{R}^n} e^{2\pi i k\cdot x - \varepsilon\pi|k|^2} \big(\hat{f}_j(k) - \hat{f}_j(k)\big) \,\mathrm{d}\mathscr{L}^n(k)\Big| \le \Big(\int_{\mathbb{R}^n} e^{-2\varepsilon\pi|k|^2} \mathrm{d}k\Big)^{1/2} \|\hat{f}_j - \hat{f}\|_2 \to 0,$$

as $j \to \infty$. So taking the limit as $j \to \infty$ in $(\star)$,

$$\int_{\mathbb{R}^n} e^{2\pi i k\cdot x - \varepsilon\pi|k|^2} \hat{f}(k) \,\mathrm{d}\mathscr{L}^n(k) = \varepsilon^{-n/2} \int_{\mathbb{R}^n} e^{-\pi|k-x|^2/\varepsilon} f(k) \,\mathrm{d}\mathscr{L}^n(k) \quad \forall\, \varepsilon > 0. \quad (*)$$

From Theorem - Approximation in L^p with $p = 2$, we know the *rhs* of $(*)$ converges to $f(x)$ in L^2. Hence there is a subsequence (that we don't rename) such that the *rhs* of $(*)$ converges to $f(x)$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$. To see that the *lhs* of $(*)$ converges (up to a subsequence) to $\hat{\hat{f}}(-x)$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$, set first $g_\varepsilon(k) := e^{-\varepsilon\pi|k|^2} \hat{f}(k)$ and observe that $g_\varepsilon \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}^n)$ for all $\varepsilon > 0$. This allows to write $lhs(*) = \hat{g}_\varepsilon(-x)$. Next we claim that g_ε converges to $\hat{f}$ in L^2. Indeed, we have

$$\|g_\varepsilon - \hat{f}\|_2^2 = \int_{\mathbb{R}^n} \big(1 - e^{-\varepsilon\pi|k|^2}\big)^2 |\hat{f}(k)|^2 \,\mathrm{d}\mathscr{L}^n(k), \quad \varepsilon > 0.$$

Clearly the integrand converges to 0 as $\varepsilon \to 0$ for $\mathscr{L}^n$-a.e. $k \in \mathbb{R}^n$, while it is also dominated by $4|\hat{f}(k)|^2$ which is summable; hence the dominated convergence theorem applies to prove the claim. Summarizing, we have $\{g_\varepsilon \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}^n)\}_{\varepsilon>0}$ such that $\lim_{\varepsilon\to 0} \|g_\varepsilon - \hat{f}\|_2 = 0$. By definition of the Fourier transform in L^2 we readily get $\hat{\hat{f}}$ is the L^2-limit of $\hat{g}_\varepsilon$; or, $\hat{\hat{f}}(-x)$ is the L^2-limit of $\hat{g}_\varepsilon(-x)$ which equals the *lhs* of $(*)$. Passing to new subsequence we get that $lhs(*)$ converges to $\hat{\hat{f}}(-x)$ for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$. ■

Wednesday, 09/12 (12:00-13:30)

THE SYMMETRIC DECREASING REARRANGEMENT, LL-§3:

Definition. Given an $\mathscr{L}^n$-measurable set X in $\mathbb{R}^n$, we say that $f : X \to \mathbb{R}$ *vanishes at infinity* provided

(a) it is $\mathscr{L}^n$-measurable function, and

(b) the level sets of $|f|$ have finite $\mathscr{L}^n$-measure; that is

$$\text{either } \mathscr{L}^n(X) < \infty, \qquad \text{or } \mathscr{L}^n(X) = \infty \text{ and } \mathscr{L}^n\big(\{x \in X \ : \ |f(x)| > t\}\big) < \infty \quad \forall\, t > 0.$$

In the case where $\mathscr{L}^n(X) = \infty$, (b) means that there cannot exist a $\delta > 0$ and a $K \Subset X$ such that $|f(x)| \geq \delta$ for all $x \in X \setminus K$. If such δ and K exist, we would have for all $t \in (0, \delta)$ that

$$\mathscr{L}^n\big(\{x \in X \ : \ |f(x)| > t\}\big) \geq \mathscr{L}^n\big(\{x \in X \ : \ |f(x)| \geq \delta\}\big) \geq \mathscr{L}^n(X \setminus K) = \infty,$$

a contradiction. This justifies the phrase "vanishing at infinity".

Notation. In what follows we write ω_n for the Lebesgue measure of a unit ball of $\mathbb{R}^n$.

Definition 1. The *symmetric rearrangement* $A^\star$, of an $\mathscr{L}^n$-measurable $A \subset \mathbb{R}^n$ is

$$A^\star := \begin{cases} \emptyset & \text{if } \mathscr{L}^n(A) = 0, \\ B_{R_A}(0) \text{ with } R_A := \big(\mathscr{L}^n(A)/\omega_n\big)^{1/n} & \text{if } \mathscr{L}^n(A) > 0, \\ \mathbb{R}^n & \text{if } \mathscr{L}^n(A) = \infty. \end{cases}$$

Note that in any case there holds $\mathscr{L}^n(A^\star) = \mathscr{L}^n(A)$.

Definition 2. The *symmetric decreasing rearrangement* $\chi_A^\star$ of the characteristic function of an $\mathscr{L}^n$-measurable $A \subset \mathbb{R}^n$ with $\mathscr{L}^n(A) < \infty$, is

$$\chi_A^\star := \chi_{A^\star}.$$

Definition 3. The *symmetric decreasing rearrangement* $f^\star$ of a function $f : X \to \mathbb{R}$ that vanishes at infinity is given by

$$\begin{aligned} f^\star(x) &:= \int_0^\infty \chi^\star_{\{z \in X \ : \ |f(z)| > r\}}(x) \, \mathrm{d}\mathscr{L}^1(r) \\ &= \int_0^\infty \chi_{\{z \in X \ : \ |f(z)| > r\}^\star}(x) \, \mathrm{d}\mathscr{L}^1(r), \qquad \text{by Definition 2.} \end{aligned}$$

Remark. Since f vanishes at infinity, the sets $A_r := \{z \in X \ : \ |f(z)| > r\}$, $r > 0$, satisfy $\mathscr{L}^n(A_r) < \infty$. Consequently, A_r, $r > 0$, satisfy the requirements of Definition 2 for defining $\chi^\star_{A_r}$, involved in Definition 3.

Remark. Compare definition 3 with the layer cake representation formula of Example 6.0.3-(i), which asserts that

$$|f(x)| := \int_0^\infty \chi_{\{z \in X \ : \ |f(z)| > r\}}(x) \, \mathrm{d}\mathscr{L}^1(r). \tag{1}$$

Proposition - Properties of $f^\star$. *Suppose $f : X \to \mathbb{R}$ vanishes at infinity. Then*

(i) *$f^\star$ is a nonnegative measurable function,*

(ii) *$f^\star$ is radially symmetric and non-increasing, that is*

$$f^\star(x) = f^\star(y) \quad \text{whenever } |x| = |y|, \qquad \text{and} \qquad f^\star(x) \geq f^\star(y) \quad \text{whenever } |x| \leq |y|,$$

Proof. Let $x, y \in \mathbb{R}^n$ be such that $|x| \ = \ (\leq) \ |y|$. Then

$$\chi_{B_R(0)}(x) \ = \ (\geq) \ \chi_{B_R(0)}(y) \quad \forall\, R \in (0,\infty).$$

In particular, since f vanishes at infinity,

$$\chi_{\{z\in X \,:\, |f(z)|>r\}^\star}(x) \ = \ (\geq) \ \chi_{\{z\in X \,:\, |f(z)|>r\}^\star}(y) \quad \forall\, r \in (0,\infty).$$

Integrating this with respect to r we deduce $f^\star(x) \ = \ (\geq) \ f^\star(y)$. ∎

(iii) *for all $t > 0$ there holds $\{z \in X \,:\, f^\star(z) > t\} = \{z \in X \,:\, |f(z)| > t\}^\star$.*

Proof. Let $x \in \{z \in X \,:\, f^\star(z) > t\}$. Assume that x is not in the ball $\{z \in X \,:\, |f(z)| > t\}^\star$, then x is also not in any concentric ball with smaller radius, that is, x is not in any ball $\{z \in X \,:\, |f(z)| > r\}^\star$ with $r > t$. Thus

$$f^\star(x) := \int_0^\infty \chi_{\{z\in X \,:\, |f(z)|>r\}^\star}(x)\, \mathrm{d}\mathscr{L}^1(r) = \int_0^t \chi_{\{z\in X \,:\, |f(z)|>r\}^\star}(x)\, \mathrm{d}\mathscr{L}^1(r) \leq t,$$

a contradiction. We proved $\{z \in X \,:\, f^\star(z) > t\} \subseteq \{z \in X \,:\, |f(z)| > t\}^\star$. Now let x be in the ball $\{z \in X \,:\, |f(z)| > t\}^\star$. The openness of the ball implies that x has to be also in some concentric ball with smaller radius; that is, there exists $\tilde{t} > 0$ such that $\tilde{t} > t$ and $x \in \{z \in X \,:\, |f(z)| > \tilde{t}\}^\star$. We readily get that x is in any concentric ball with larger radius; that is x is in any ball $\{z \in X \,:\, |f(z)| > r\}^\star$ with $r < \tilde{t}$. Thus

$$f^\star(x) := \int_0^\infty \chi_{\{z\in X \,:\, |f(z)|>r\}^\star}(x)\, \mathrm{d}\mathscr{L}^1(r) \geq \int_0^{\tilde{t}} \chi_{\{z\in X \,:\, |f(z)|>r\}^\star}(x)\, \mathrm{d}\mathscr{L}^1(r) = \tilde{t} > t,$$

establishing the reverse inclusion $\{z \in X \,:\, |f(z)| > t\}^\star \subseteq \{z \in X \,:\, f^\star(z) > t\}$. ∎

Remark. From the last property we deduce

$$\mathscr{L}^n\big(\{z \in X \,:\, |f(z)| > t\}^\star\big) = \mathscr{L}^n\big(\{z \in X \,:\, f^\star(z) > t\}\big) \qquad \forall\, t > 0,$$

and recalling Definition 1, we further obtain the so called *equimeasurability* property

$$\mathscr{L}^n\big(\{z \in X \,:\, |f(z)| > t\}\big) = \mathscr{L}^n\big(\{z \in X \,:\, f^\star(z) > t\}\big) \qquad \forall\, t > 0.$$

Using this we easily get that symmetric decreasing rearrangement preserves the L^p-norm for any $p \in [1,\infty]$. To see this for $p \in [1,\infty)$, we take the layer cake representation formula in the form of Example 6.0.3-(ii) to write

$$\begin{aligned} \|f\|_p^p &= p\int_0^\infty r^{p-1}\mathscr{L}^n\big(\{z \in X \,:\, |f(z)| > r\}\big)\mathrm{d}\mathscr{L}^1(r) \\ &= p\int_0^\infty r^{p-1}\mathscr{L}^n\big(\{z \in X \,:\, |f(z)| > r\}^\star\big)\mathrm{d}\mathscr{L}^1(r) \\ &= p\int_0^\infty r^{p-1}\mathscr{L}^n\big(\{z \in X \,:\, f^\star(z) > r\}\big)\mathrm{d}\mathscr{L}^1(r) = \|f^\star\|_p^p. \end{aligned}$$

Theorem (proved) - Hardy-Littlewood inequality. *Suppose that $f, g : X \to \mathbb{R}$ vanish at infinity. Then*

$$\int_X |f||g|\, \mathrm{d}\mathscr{L}^n \leq \int_X f^\star g^\star\, \mathrm{d}\mathscr{L}^n.$$

Proof. Write $A_r := \{z \in X \ : \ |f(z)| > r\}$, $r > 0$. By the layer cake representation formula and the Fubini theorem we have

$$\begin{aligned}\int_X |f||g| \, \mathrm{d}\mathscr{L}^n &= \int_X \int_0^\infty \chi_{A_r}(x) \, \mathrm{d}\mathscr{L}^1(r) \int_0^\infty \chi_{A_s}(x) \, \mathrm{d}\mathscr{L}^1(s) \, \mathrm{d}\mathscr{L}^n(x) \\ &= \int_0^\infty \int_0^\infty \int_X \chi_{A_r}(x)\chi_{A_s}(x) \, \mathrm{d}\mathscr{L}^n(x) \, \mathrm{d}\mathscr{L}^1(r) \, \mathrm{d}\mathscr{L}^1(s) \\ &= \int_0^\infty \int_0^\infty \mathscr{L}^n(A_r \cap A_s) \, \mathrm{d}\mathscr{L}^1(r) \, \mathrm{d}\mathscr{L}^1(s).\end{aligned}$$

On the other hand, by Definition 3 and the Fubini theorem we have

$$\begin{aligned}\int_X f^\star g^\star \, \mathrm{d}\mathscr{L}^n &= \int_X \int_0^\infty \chi_{A_r^\star}(x) \, \mathrm{d}\mathscr{L}^1(r) \int_0^\infty \chi_{A_s^\star}(x) \, \mathrm{d}\mathscr{L}^1(s) \, \mathrm{d}\mathscr{L}^n(x) \\ &= \int_0^\infty \int_0^\infty \int_X \chi_{A_r^\star}(x)\chi_{A_s^\star}(x) \, \mathrm{d}\mathscr{L}^n(x) \, \mathrm{d}\mathscr{L}^1(r) \, \mathrm{d}\mathscr{L}^1(s) \\ &= \int_0^\infty \int_0^\infty \mathscr{L}^n(A_r^\star \cap A_s^\star) \, \mathrm{d}\mathscr{L}^1(r) \, \mathrm{d}\mathscr{L}^1(s).\end{aligned}$$

This shows it is enough to prove that $\mathscr{L}^n(A_r \cap A_s) \leq \mathscr{L}^n(A_r^\star \cap A_s^\star)$ for all $r, s \in (0, \infty)$, or even that $\mathscr{L}^n(A \cap B) \leq \mathscr{L}^n(A^\star \cap B^\star)$ for all $\mathscr{L}^n$-measurable $A, B \subset \mathbb{R}^n$ such that $\mathscr{L}^n(A), \mathscr{L}^n(B) < \infty$. But this is true since, if for example $\mathscr{L}^n(A) \leq \mathscr{L}^n(B)$, then $A^\star \subseteq B^\star$ and so $\mathscr{L}^n(A^\star \cap B^\star) = \mathscr{L}^n(A^\star) = \mathscr{L}^n(A) \geq \mathscr{L}^n(A \cap B)$. ■

Appilcation #1 (proved). The last remark together with the above theorem easily imply that distance in L^2 does not increase after taking symmetric decreasing rearrangements of functions, that is

$$\|f^\star - g^\star\|_2 \leq \|f - g\|_2.$$

Friday, 11/12 (12:00-13:30)

Application #2 (proved). If $U \subset \mathbb{R}^n$ is $\mathscr{L}^n$-measurable with $\mathscr{L}^n(U) < \infty$, then

$$\sup_{x \in U} \int_U \frac{1}{|x-z|^\sigma} \mathrm{d}z \leq \frac{\omega_n^{\sigma/n}}{1 - \sigma/n} \left[\mathscr{L}^n(U)\right]^{1-\sigma/n} \quad \text{for all } 0 < \sigma < n.$$

Proof. Given $x \in U$ write $U_x := \{x - z \mid z \in U\}$. Then,

$$\begin{aligned}\int_U \frac{1}{|x-z|^\sigma} \mathrm{d}z = \int_{U_x} \frac{1}{|y|^\sigma} \mathrm{d}y &= \int_{\mathbb{R}^n} \chi_{U_x}(y) \frac{1}{|y|^\sigma} \mathrm{d}y \\ &\leq \int_{\mathbb{R}^n} \chi_{U_x}^\star(y) \Big(\frac{1}{|y|^\sigma}\Big)^\star \mathrm{d}y = \int_{\mathbb{R}^n} \chi_{U_x^\star}(y) \frac{1}{|y|^\sigma} \mathrm{d}y = \int_{U_x^\star} \frac{1}{|y|^\sigma} \mathrm{d}y,\end{aligned}$$

where we have used the Hardy-Littlewood inequality to pass to the second line. But $U_x^\star$ is a ball centered at 0 and volume equal to $\mathscr{L}^n(U_x)$. Since the translation invariance of $\mathscr{L}^n$ implies $\mathscr{L}^n(U_x) = \mathscr{L}^n(U)$, we obtain $U_x^\star = B_{R_U}(0)$ where $R_U := \big(\mathscr{L}^n(U)/\omega_n\big)^{1/n}$. Therefore,

$$\int_U \frac{1}{|x-z|^\sigma} \mathrm{d}z \leq \int_{B_{R_U}(0)} \frac{1}{|y|^\sigma} \mathrm{d}y = \int_0^{R_U} \int_{\partial B_r(0)} \frac{1}{|y|^\sigma} \mathrm{d}S_y \, \mathrm{d}r = n\omega_n \int_0^{R_U} r^{-\sigma+n-1} \, \mathrm{d}r = \frac{n\omega_n}{n-\sigma} R_U^{n-\sigma},$$

and substituting R_U by $\big(\mathscr{L}^n(U)/\omega_n\big)^{1/n}$ gives the result. ■

Theorem - Riesz rearrangement inequality. *Suppose that $f, g, h : \mathbb{R}^n \to \mathbb{R}$ vanish at infinity. Then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x)||g(x-y)||h(y)| \mathrm{d}\mathscr{L}^n(x) \mathrm{d}\mathscr{L}^n(y) \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^\star(x) g^\star(x-y) h^\star(y) \mathrm{d}\mathscr{L}^n(x) \mathrm{d}\mathscr{L}^n(y).$$

Application #1 (proved). Among all homogeneous 3-dimensional bodies, whose volume and density are fixed, the ball generates the gravitational field having the largest energy.

Application #2 Theorem - Hardy-Littlewood-Sobolev inequality (proved) *Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, $n \in \mathbb{N}$, where $p, q > 1$ are such that $1 < 1/p + 1/q < 2$. Then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)||g(y)|}{|x-y|^\sigma} \mathrm{d}\mathscr{L}^n(x) \mathrm{d}\mathscr{L}^n(y) \le C(n,p,q)\|f\|_p\|g\|_q, \qquad \textit{where } \sigma = n\Big(\frac{p-1}{p} + \frac{q-1}{q}\Big).$$

Remarks. Functions in $L^p(\mathbb{R}^n)$ necessarily vanish at infinity. Also the assumption $1 < 1/p + 1/q < 2$ implies $0 < \sigma < n$ as required in Application 2 of the Hardy-Littlewood inequality.

Lemma (proved). *For any $u \in C^1_c(\mathbb{R}^n)$ we have*

$$u(y) = \frac{1}{n\omega_n}\int_{\mathbb{R}^n} \frac{(y-x)\cdot\nabla u(x)}{|y-x|^n}\,\mathrm{d}x \qquad \forall\, y \in \mathbb{R}^n.$$

Proof. Using polar coordinates around x and then changing variables by $x = y + rz$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{(y-x)\cdot\nabla u(x)}{|y-x|^n}\,\mathrm{d}x &= \int_0^\infty \int_{\partial B_r(x)} \frac{(y-x)\cdot\nabla u(x)}{|y-x|^n}\,\mathrm{d}S_x\,\mathrm{d}r \\
&= -\int_0^\infty \int_{\partial B_1(0)} z\cdot\nabla u(y+rz)\,\mathrm{d}S_z\,\mathrm{d}r \\
&= -\int_{\partial B_1(0)} \int_0^\infty z\cdot\nabla u(y+rz)\,\mathrm{d}r\,\mathrm{d}S_z \\
&= -\int_{\partial B_1(0)} \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}r}\big[u(y+rz)\big]\,\mathrm{d}r\,\mathrm{d}S_z = -\int_{\partial B_1(0)} -u(y)\,\mathrm{d}S_z = n\omega_n u(y),
\end{aligned}$$

where we have also used Fubini's Theorem. ■

Theorem (proved) - L^p-Sobolev inequality. The Hardy-Littlewood-Sobolev inequality implies the *L^p-Sobolev inequality*; that is

$$\|u\|_{p_S} \le c(n,p)\||\nabla u|\|_p \qquad \forall\, u \in C^1_c(\mathbb{R}^n), \qquad \textit{where } p_S := np/(n-p),\ 1 < p < n.$$

Proof. From the above lemma we get

$$|u(y)| \le \frac{1}{n\omega_n}\int_{\mathbb{R}^n} \frac{|\nabla u(x)|}{|x-y|^{n-1}}\,\mathrm{d}x \qquad \forall\, y \in \mathbb{R}^n.$$

Hence, for any $\mathscr{L}^n$-measurable function g,

$$\int_{\mathbb{R}^n} |u(y)||g(y)| \mathrm{d}\mathscr{L}^n(y) \le \frac{1}{n\omega_n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\nabla u(x)||g(y)|}{|x-y|^{n-1}}\,\mathrm{d}x\,\mathrm{d}\mathscr{L}^n(y).$$

Since $|\nabla u| \in C_c(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ we can apply the H-L-S inequality with $\sigma = n-1$, provided $g \in L^q(\mathbb{R}^n)$ with $q := p_S' = p_S/(p_S - 1)$. Thus,

$$\int_{\mathbb{R}^n} |u||g| \mathrm{d}\mathscr{L}^n \le n\omega_n C(n,p)\||\nabla u|\|_p\|g\|_{p_S'} \qquad \forall\, g \in L^{p_S'}(\mathbb{R}^n),$$

from which we further obtain

$$\sup_{\substack{g\in L^{p_S'}(\mathbb{R}^n)\\ \|g\|_{p_S'}\le 1}} \Big|\int_{\mathbb{R}^n} ug\,\mathrm{d}\mathscr{L}^n\Big| \le n\omega_n C(n,p)\||\nabla u|\|_p.$$

Now notice that the left hand side of this is precisely $\|u\|_{p_S}$. Indeed, since $u\in C_c^1(\mathbb{R}^n)\subset L^{p_S}(\mathbb{R}^n)$ defines a bounded linear functional ℓ_u of $L^{p_S'}(\mathbb{R}^n)$ through

$$\ell_u(g) := \int_{\mathbb{R}^n} ug\,\mathrm{d}\mathscr{L}^n \qquad g\in L^{p_S'}(\mathbb{R}^n),$$

we know from "the dual of L^p"-Theorem that $\|\ell_u\| = \|u\|_{p_S}$, where

$$\|\ell_u\| := \sup_{\substack{g\in L^{p_S'}(\mathbb{R}^n)\\ \|g\|_{p_S'}\le 1}} |\ell_u(g)|$$

■

Monday, 14/12 (10:15-11:45)

Theorem - Logarithmic Sobolev inequality - L.L., Theorem 8.14. The L^2-*Sobolev inequality* implies the logarithmic Sobolev inequalty; that is, for all $a>0$ there holds

$$\frac{a^2}{\pi}\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x \ge \int_{\mathbb{R}^n}\log(u^2)u^2\mathrm{d}x + n\big(1+\log a\big) \qquad \forall\, u\in C_c^1(\mathbb{R}^n) \text{ with } \|u\|_2 = 1.$$

Proof. Given $u\in C_c^1(\mathbb{R}^n)$ with $\|u\|_2=1$, apply Jensen's inequality with $\mu := u^2\mathscr{L}^n$ for the concave function $f(t)=\log t$, $t\ge 0$, as follows

$$\frac{2}{n}\int_{\mathbb{R}^n}\log(u^2)u^2\mathrm{d}x = \frac{2}{2_S}\int_{\mathbb{R}^n}\log\big((u^2)^{2_S/n}\big)\mathrm{d}\mu \le \frac{2}{2_S}\log\Big(\int_{\mathbb{R}^n}\big((u^2)^{2_S/n}\big)\mathrm{d}\mu\Big) = \log\big(\|u\|_{2_S}^2\big)$$

From the L^2-Sobolev inequality we further get

$$\frac{2}{n}\int_{\mathbb{R}^n}\log(u^2)u^2\mathrm{d}x \le \log\big(\kappa(n)\||\nabla u|\|_2^2\big). \tag{3}$$

This is already a type of logarithmic Sobolev inequality. Its best constant is known[4] to be

$$\kappa(n) = \frac{2}{ne\pi}.$$

Applying $\log B \le \frac{B-A}{A} + \log A$ for all $A,B>0$, with

$$B := \kappa(n)\||\nabla u|\|_2^2 \qquad \text{and} \qquad A := \frac{n\pi\kappa(n)}{2a^2},\ a>0,$$

we deduce from (3) the logarithmic Sobolev inequality

$$\frac{a^2}{\pi}\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x \ge \int_{\mathbb{R}^n}\log(u^2)u^2\mathrm{d}x + n\Big(1+\log a - \frac{1}{2}\log\big(ne\pi\kappa(n)/2\big)\Big).$$

Note that to get the inequality as stated in the theorem we need to prove (3) with the best constant. ■

We had an overview of what we have learned in this course.

[4] see WEISSLER, F. B. *Logarithmic Sobolev inequalities for the heat-diffusion semigroup*. Trans. Amer Math. Soc. **237** (1978), 255-269.