Partial Differential Equations

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Sheet 1

11.2.2019

1. The linear transport equation

Let $b \in \mathbb{R}^n$. The (homogeneous) linear transport equation with direction b is given by the following partial differential equation of first order:

$$\dot{u} + b \cdot \nabla u = 0 \, . \tag{$*$}$$

Here, $u = u(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is the sought solution function, $\dot{u}$ denotes the derivative of u with respect to $t \in \mathbb{R}$ and the gradient ∇u is considered with respect to $x \in \mathbb{R}^n$.

(a) Prove: If $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a solution of $(*)$, then u is constant on each of the parallel lines with direction $(b,1) \in \mathbb{R}^n \times \mathbb{R}$. *(4 points)*

(b) Let $g \in C^1(\mathbb{R}^n)$. Prove that $u(x,t) := g(x - tb)$ is the *unique* solution of $(*)$ satisfying $u(\cdot, 0) = g$. *(5 points)*

2. Laplacian and Laplace equation

(a) Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $f \in C^2(\Omega)$ and $g \in C^1(\Omega)$. Show that

$$g \triangle f = \nabla \cdot (g \nabla f) - \nabla f \cdot \nabla g$$

holds. *(4 points)*

(b) Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain. Let $u \in C^2(\overline{\Omega})$ be a solution of the *boundary value problem*

$$\triangle u = 0 \quad \text{with} \quad u|_{\partial\Omega} = 0 \, .$$

Show $u \equiv 0$. *(5 points)*

[Hint: Investigate $\int_\Omega u(\triangle u) \mathrm{d}\mu$ with the help of task 2(a) und the Gauss' divergence theorem.]

Turn the page, please.

3. Extremals of the Dirichlet integral

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. Let $u \in C^2(\overline{\Omega})$ minimize the Dirichlet integral $\int_\Omega \|\nabla u\|^2$ in $C^1(\overline{\Omega})$ with fixed boundary values, i.e. for each function $v \in C^1(\overline{\Omega})$ with $v|_{\partial\Omega} = u|_{\partial\Omega}$, we have

$$\int_\Omega \|\nabla u\|^2 \leq \int_\Omega \|\nabla v\|^2 .$$

The aim of this exercise is to show that u is harmonic. Thereto, we proceed as follows:

Let an arbitrary $\lambda \in C^1(\overline{\Omega})$ be given whose support $\operatorname{supp}(\lambda) := \overline{\{ x \in \overline{\Omega} \,|\, \lambda(x) \neq 0 \}}$ is contained in Ω. We consider the family of functions $(u_t)_{t \in \mathbb{R}}$ with

$$u_t : \overline{\Omega} \to \mathbb{R},\ x \mapsto u(x) + t\lambda(x)$$

as well as the differentiable function

$$f : \mathbb{R} \to \mathbb{R},\ t \mapsto \int_\Omega \|\nabla u_t\|^2 .$$

Now proceed as follows.

(a) Show $\frac{\partial}{\partial t}\big|_{t=0} (\|\nabla u_t\|^2) = 2\nabla u \cdot \nabla\lambda$. *(4 points)*

(b) Show with the help of (a) that $f'(0) = -2\int_\Omega \lambda \triangle u$. *(4 points)*

[Hint: Task 2(a) and the Gauss' divergence theorem.]

(c) On the other hand, deduce from the minimal property of u that $f'(0) = 0$. *(2 points)*

(d) Combine (b) and (c) to deduce that $\triangle u|_\Omega = 0$. Hence, u is harmonic.

Here, you may use without proof that for each $x \in \Omega$ and for each open neighbourhood $U \subset \Omega$ of x, there is a function $\lambda \in C^1(\overline{\Omega})$ satisfying $\lambda \geq 0$, $\operatorname{supp}(\lambda) \subset U$ und $\lambda(x) = 1$. Apply (b) and (c) to such a λ in order to prove that $\triangle u(x) = 0$. *(4 points)*

This sheet has to be handed until **18.2.2019, 4 pm** into the respective mail box at the C-entrance of the A5 building.

Note: The new exercise sheets will be available on Mondays on the website of the chair "Lehrstuhl Mathe III"

`http://analysis.math.uni-mannheim.de`→Lehre→FS 2019 → Partial Differential Equations.

The sheets have to be handed until the respective next Monday into the corresponding mail box "Partial Differential Equations" at the entrance of the A5 building "C-Teil". In order to be admitted to the exam at the end of the term, it is sufficient to gain at least 50 % of the total sum of points in the exercise sheets. For any further questions concerning the exercises and tutorials, you can contact Volker Eing. E-Mail: v.eing@math.uni-mannheim.de).