Introduction to Partial Differential Equations

Martin Schmidt
Volker Eing

Exercise sheet 2

September 10th, 2018

(submit your solutions until September 17th, 4pm in A5, C, east entrance, post box: 46236)

5. An Induced Distribution.

Let $F \in \mathcal{D}'(\mathbb{R}^{n+m})$ and $\psi \in C_0^\infty(\mathbb{R}^m)$. Define

$$G : C_0^\infty(\mathbb{R}^n) \to \mathbb{R},$$
$$\varphi \mapsto F(\varphi \times \psi).$$

Show that G is a distribution on $C_0^\infty(\mathbb{R}^n)$, i.e. $G \in \mathcal{D}'(\mathbb{R}^n)$.
(*Caution:* Don't forget to show that the above definition makes sense, in the sense that $\varphi \times \psi$ is indeed in $C_0^\infty(\mathbb{R}^{m+n})$!) *(10 points)*

6. Derivatives And Distributions.

Let $n \in \mathbb{N}$, $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ and let (x,t) with $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ denote the elements in $\mathbb{R}^n \times \mathbb{R}$. Show that, $\partial_t F = 0$ if and only if there is a distribution $G \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$F(\varphi) = G\Big(\int_{-\infty}^{\infty} \varphi(-,t)dt\Big).$$

In order to show that prove the following steps:

(a) Define

$$\mathcal{I} : \mathcal{D}(\mathbb{R}^n \times \mathbb{R}) \to \mathcal{D}(\mathbb{R}^n),$$
$$\varphi \mapsto \Big(x \mapsto \int_{-\infty}^{\infty} \varphi(x,t)dt\Big).$$

Show that $\mathcal{I}$ indeed maps to $\mathcal{D}(\mathbb{R}^n)$, is continous and linear.

(b) Show that for $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}), \partial_t F = 0$ if and only if $F \equiv 0$ on the kernel of $\mathcal{I}$.

(c) Finally show the statement by showing that $\partial_t F = 0$ if and only if there exists a $G \in \mathcal{D}'(\mathbb{R}^n)$ with $F(\varphi) = G(\mathcal{I}(\varphi))$.

(d) Now that you have shown the statement, for $i \in \{1, \dots, n\}$ show that $\partial_i \mathcal{I}(\varphi) = \mathcal{I}(\partial_i \varphi)$.

(e) Use (c) to show that for any $F \in \mathcal{D}'(\mathbb{R}^n)$ with $\partial_i F = 0$ for all $i \in \{1, \dots, n\}$ there exists $\lambda \in \mathbb{R}$, such that F corresponds to the constant function $\lambda \cdot 1$. And give an explicit formula for λ in terms of F. *(6 points each)*

7. Now it's your work.

In the proof of Lemma 1.9 there are two statements which you shall prove:

(a) "Due to the continuity of F with respect to the semi norms $||\cdot||_{K,0}$ the functions $x \mapsto F(\mathbf{T}_x \mathbf{P} g)$ are continous."

(b) "[...] these functions are smooth since $\frac{\mathbf{T}(y+\epsilon h) - \mathbf{T}(y)}{\epsilon}\phi$ converges for all $\phi \in C^\infty(\mathbb{R}^n)$ in the limit $\epsilon \to 0$ on the space $C^\infty(\mathbb{R}^n)$ with respect to the topology induced by the semi norms $||\cdot||_{K,\alpha}$ to $\mathbf{T}(y)(\sum_{i=1}^n h_i \partial_i \phi)$. *(5 points each)*