Introduction to Partial Differential Equations

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Exercise sheet 6

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(submit your solutions until Wednesday, October 17th, 4pm in A5, C, east entrance, post box: 46236)

17. Harmonic Functions on $B(0,1) \subset \mathbb{R}^2$.

Let $B(0,1) := \{ (x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1 \}$ be the open unit disc in $\mathbb{R}^2$.

(a) Let $u \in C^2(\overline{B(0,1)})$ be a harmonic function on $B(0,1)$ which is given in polar coordinates $u = u(r,\varphi)$ (with $0 \le r \le 1$ and $-\pi < \varphi \le \pi$). Show that in this coordinates

$$\int_{\partial B(0,1)} \frac{\partial u}{\partial r}(x)\, \mathrm{d}\sigma(x) = 0$$

(10 points)

[*hint:* $0 = \int_{B(0,1)} \triangle u = \dots$]

(b) "Guess" a solution for each of the following $u \in C^2(\overline{B(0,1)})$ *Neumann-Problems* or show that there is no solution.

(i) $\triangle u = 0$ on $B(0,1)$ with $\frac{\partial u}{\partial r} = \sin(\varphi)$ on $\partial B(0,1)$. *(4 points)*

(ii) $\triangle u = 0$ on $B(0,1)$ with $\frac{\partial u}{\partial r} = \sin^2(\varphi)$ on $\partial B(0,1)$. *(4 points)*

18. Harmonic Polynomials.

Let $n \in \mathbb{N}$ and $d \in \mathbb{N}_0$. A *real homogeneous polynomial of degree* d *on* $\mathbb{R}^n$ is a linear combination of *monomials* of the form $Q := x_1^{d_1} \cdot x_2^{d_2} \cdot \ldots \cdot x_n^{d_n}$ with $d_k \in \mathbb{N}_0$ and $d_1 + \ldots + d_n = d$; for each such monomial Q we define $m(Q) := \max\{d_1, \ldots, d_n\}$.

The vector space of real homogeneous polynomials of degree d on $\mathbb{R}^n$ is denoted by $\mathcal{P}(d,n)$. We want to determine the dimension of the subspace

$$\mathcal{H}(d,n) := \{ P \in \mathcal{P}(d,n) \,|\, \triangle P = 0 \}$$

of harmonic polynomials in $\mathcal{P}(d,n)$:

(a) Show, using combinatorics, that $\dim \mathcal{P}(d,n) = \binom{n+d-1}{d}$.

(4 points)

(b) show:

$$\triangle[\mathcal{P}(d,n)] \begin{cases} = \{0\} & \text{for } d \in \{0,1\} \\ \subset \mathcal{P}(d-2,n) & \text{for } d \ge 2 \end{cases} .$$

(5 points)

(c) Show, that in the case $d \ge 2$ the linear map $\triangle : \mathcal{P}(d,n) \to \mathcal{P}(d-2,n)$ is surjective.

(10 points)

[*Hint:* It suffices to show that the monomials $Q \in \mathcal{P}(d-2,n)$ are in the image of $\triangle|_{\mathcal{P}(d,n)}$. Show the following "finite induction in $m(Q)$", i.e.:

(i) <u>Induction basis</u>:
Every monomial $Q \in \mathcal{P}(d-2,n)$ with $m(Q) = d-2$ is in the image of $\triangle|_{\mathcal{P}(d,n)}$.

(ii) Induction step:
Let $M \in \{1, \ldots, d-2\}$. If all $Q \in \mathcal{P}(d-2, n)$ with $m(Q) \in \{M+1, \ldots, d-2\}$ are in the image of $\triangle|_{\mathcal{P}(d,n)}$, then also every monomial $Q \in \mathcal{P}(d-2, n)$ with $m(Q) = M$.]

(d) combine (a) to (c) to determine $\dim \mathcal{H}(d, n)$. *(5 points)*

19. About the Maximum Principle of harmonic functions.

Let $\Omega \subset \mathbb{R}^n$ be an open, connected and bounded subset and $f : \Omega \to \mathbb{R}$ and $g_1, g_2 : \partial\Omega \to \mathbb{R}$ continous functions. Let also $u_1, u_2 : \overline{\Omega} \to \mathbb{R}$ two continous, and inside Ω twice continously differentiable solutions of the following Dirichlet-problem

$$-\triangle u_k|\Omega = f\,, \qquad u_k|\partial\Omega = g_k$$

für $k \in \{1, 2\}$.

Show: If $g_1 \leq g_2$, then $u_1 \leq u_2$. *(8 points)*