Introduction to Partial Differential Equations

Martin Schmidt **Exercise sheet 3** September 19th, 2018

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(submit your solutions until Wednesday, September 26th, 4pm in A5, C, east entrance, post box: 46236)

8. Another detail of Lemma 1.9.

Let $\phi \in C_0^\infty(\mathbb{R}^n)$ and $F \in \mathcal{D}'(\mathbb{R}^n)$. The aim of this exercise is to show the following identity from Lemma 1.9

$$F(\mathrm{P}g * \phi) = \int_{\mathbb{R}^n} F(\mathrm{T}_x \mathrm{P}g)\phi(x)\mathrm{d}^n x.$$

For that show the following steps:

(a) Let $\varepsilon > 0$. Show that there exist finitely many pairs $(\phi_{\varepsilon,i}, x_{\varepsilon,i})_{i=1,\dots k_\varepsilon}$ in $\mathbb{R} \times \mathbb{R}^n$, such that for all $g \in C_0^\infty(\mathcal{R}^n)$

$$\left|\left|\phi * g - \sum_{i=1}^{k_\varepsilon} \phi_{\varepsilon,i}\mathrm{T}_{x_{\varepsilon,i}} g\right|\right|_\infty \leq C \cdot \varepsilon.$$

Here $C = C(\phi, g)$ is a constant depending on ϕ and g but not depending on ε.

[*Hint:* Since ϕ has compact support, there is a finite cover of $K := \mathrm{supp}(\phi)$ by balls of radius ε. Furthermore there is -without proof- a *partition of unity*, which means that there are functions $\psi_1, \dots, \psi_{k_\varepsilon} \in C_0^\infty(\mathbb{R}^n)$, such that $0 \leq \psi_i \leq 1$, $\mathrm{supp}(\psi_i) \subset U_i$ and $\sum_{i=1}^{k_\varepsilon} \psi|K = 1|K$. With the help of this data set $\phi_{\varepsilon,i} := \phi(x_{\varepsilon,i}) \cdot \int_{U_i} \psi_i(y)\mathrm{d}^n y$. Now use that ψ and g are Lipschitz-continous (because they are C^1-functions with compact support) to show the estimate.] *(10 points)*

(b) For every $g \in C_0^\infty(\mathbb{R}^n)$ and for every Multiindex $\alpha \in \mathbb{N}_0^n$ show that

$$\lim_{n\to\infty} \left|\left|\phi * g - \sum_{i=1}^{k_{1/n}} \phi_{1/n,i}\mathrm{T}_{x_{1/n,i}} g\right|\right|_{K,\alpha} = 0,$$

where $K \subset \mathbb{R}^n$ is any compact set.

[*Hint:* Show first, using (a) the case $\alpha = 0$. Then deduce the general case by using $\partial^\gamma g$ instead of g for any multiindex $0 \leq \gamma \leq \alpha$. For that you need also $\partial^\gamma(f*g) = f*\partial^\gamma g = \partial^\gamma f*g$ (show that too!).] *(8 points)*

(c) Following (b) show that,

$$\lim_{n\to\infty} \sum_{i=1}^{k_{1/n}} \phi_{1/n,i} F(\mathrm{T}_{x_{1/n,i}} g) = F(\phi * g).$$

(3 points)

(d) Finally show the statement from the lecture by using (c).

[*Hint:* use the dominated convergence theorem] *(5 points)*

9. Some examples of Distributions.

(a) Show that

$$F : C_0^\infty(\mathbb{R}) \to \mathbb{R}$$
$$\phi \mapsto \int_{\mathbb{R}} x^3 \cdot \phi''(x)\mathrm{d}x$$

is a distribution on $\mathbb{R}$ and find a function $f : \mathbb{R} \to \mathbb{R}$ with

$$F(\phi) = \int_{\mathbb{R}} f(x) \cdot \phi(x)\mathrm{d}x \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}).$$

(3 points)

(b) Show, that the Dirac-Distribution

$$\delta : C_0^\infty(\mathbb{R}) \to \mathbb{R}$$
$$\phi \mapsto \phi(0)$$

is indeed a distribution on $\mathbb{R}$ and show that there is no continous function $g : \mathbb{R} \to \mathbb{R}$ with

$$\delta(\phi) = \int_{\mathbb{R}} g(x) \cdot \phi(x)\mathrm{d}x \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}).$$

(4 points)

(c) Calculate the derivatives of both F from (a) and δ from (b) in the distributional sense.

(1 point each)

10. Initial Value Problem of the Weak Transport Equation

(a) Let $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ be a weak solution of the homogeneous transport equation, i.e. $\partial_t F + b \cdot \nabla F = 0$. Define

$$\tilde{F} : C_0^\infty(\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}$$
$$\varphi \mapsto F(\tilde{\varphi})$$

where $\tilde{\varphi}(x,t) := \varphi(x + bt, t)$ for a function $\varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R})$. Show that $\tilde{F}$ is a distribution and solves $\partial_t \tilde{F} = 0$.

(b) Now use exercise 6 to show that there is a weak solution of the following Initial Value Problem in the following sense:
Let $G \in \mathcal{D}'(\mathbb{R}^n)$. There is a unique $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$, such that

(i) F is a solution of the homogeneous transport equation

$$\partial_t F + b \cdot \nabla F = 0.$$

(ii) For a sequence of mollifiers $\lambda_\varepsilon \in C_0^\infty(\mathbb{R})$ and $\phi \in C_0^\infty(\mathcal{R}^n)$ the limit $\lim_{\varepsilon\to 0} F(\phi \times \lambda_\varepsilon)$ exists and the following identity holds:

$$\lim_{\varepsilon \to 0} F(\phi \times \lambda_\varepsilon) = G(\phi).$$

(15 points in total)