Introduction to Partial Differential Equations

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(submit your solutions until Wednesday, October 10th, 4pm in A5, C, east entrance, post box: 46236)

13. Preparing the Mean Value Theorem.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continous function, $x_0 \in \mathbb{R}^n$ and $\partial B(x_0, r) := \{x \in \mathbb{R}^n \mid \|x - x_0\| = r\}$ for $r > 0$. Show that the function

$$\Phi(r) := \frac{1}{\sigma(\partial B(x_0, r))} \int_{\partial B(x_0,r)} f(x) d\sigma(x).$$

converges in the limit $r \to 0$ to $f(x_0)$. *(5 points)*

14. Fundamental solution of the Laplace-Equation.

Let $n \geq 2$.

(a) Let $u \in C^2(\mathbb{R}^n)$ be rotational symmetric, i.e. $u(x) = v(\|x\|)$, where $v : [0, \infty) \to \mathbb{R}$ is a two times continously differentiable function. Show that

$$\Delta u(x) = \|x\|^{1-n} \cdot \left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=\|x\|} (r^{n-1} \cdot v'(r)).$$

(4 points)

(b) Let $\omega_n := \int_{\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}} 1 \, \mathrm{d}^n x$ be the volume of the unit ball in $\mathbb{R}^n$. Show that the function

$$\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}, \quad x \mapsto \begin{cases} -\frac{1}{2\pi} \log(\|x\|) & \text{for} \quad n = 2 \\ \frac{1}{n(n-2)\omega_n} \|x\|^{2-n} & \text{for} \quad n \geq 3 \end{cases}$$

is harmonic. Φ is called the *fundamental solution of the Laplace-Equation.*
Show also that

$$\nabla \Phi = -\frac{1}{n\omega_n} \frac{x}{\|x\|^n}.$$

(6 points)

15. Solution of the Poisson-Equation.

Let $n \geq 2$ and Φ the fundamental solution of the Laplace-Equation from exercise 14. Consider also a two times continously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ with compact support. In this exercise we will show that the convolution of f with Φ is a solution of the Poisson-Equation $-\Delta u = f$ in $\mathbb{R}^n$.

(a) Show that the integral $\int_{\mathbb{R}^n} f(x - y)\Phi(y)\mathrm{d}^n y$ is well defined, although Φ is not defined in 0 and show that the convolution $u : \mathbb{R}^n \to \mathbb{R}$ with $u(x) := (f * \Phi)(x)$ is two times differentiable and that

$$\Delta u = \int_{\mathbb{R}^n} \Delta f(x - y) \cdot \Phi(y) \mathrm{d}^n y.$$

(4 points)

Let now $\varepsilon > 0$ We decompose the above integral Δu into two parts:

$$I_\varepsilon := \int_{B(0,\varepsilon)} \Delta f(x-y) \cdot \Phi(y) \mathrm{d}^n y,$$
$$J_\varepsilon := \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Delta f(x-y) \cdot \Phi(y) \mathrm{d}^n y.$$

(b) Show that $\lim_{\varepsilon \to 0+} I_\varepsilon = 0$. *(6 points)*

(c) Show that

$$J_\varepsilon = - \int_{\partial B(0,\varepsilon)} f(x-y) \cdot \nabla_y \Phi(y) \cdot N \mathrm{d}\sigma(y) + L_\varepsilon,$$

where L_ε is an expression which converges to 0 as ε converges to 0. *(6 points)*
(*Hint:* Use the divergence theorem, exercise 2(b) and 14(b).)

(d) Show that the expression $\int_{\partial B(0,\varepsilon)} f(x-y) \cdot \nabla_y \Phi(y) \cdot N \mathrm{d}\sigma(y)$ is the mean value of f in $\partial B(x,\varepsilon)$ and conclude that $-\triangle u = f$. *(4 points)*
(*Hint:* Use the expression for $\nabla\Phi$ from exercise 14(b). Also use the fact that the inner pointing normal unit vector field N at $\partial B(0,\varepsilon)$ is given by $N(x) = -\frac{x}{\|x\|}$ and that the volume σ_n of the unit sphere $\{ x \in \mathbb{R}^n \mid \|x\| = 1 \}$ is given by $\sigma_n = n \cdot \omega_n$. In order to show the second part use exercise 13.)

16. Subharmonic Functions.

Let $\Omega \subset \mathbb{R}^n$ be an open, connected region. A two times continously differentiable function $v : \overline{\Omega} \to \mathbb{R}$ is called *subharmonic*, if $-\triangle v \leq 0$ in Ω.

(a) Let $v : \overline{\Omega} \to \mathbb{R}$ be subharmonic. Show that for all $x \in \Omega$ and $r > 0$ with $B(x,r) \subset \Omega$:

$$v(x) \leq \frac{1}{r^{n-1}\, n\, \omega_n} \int_{\partial B(x,r)} v(y)\, d\sigma(y)\ .$$

(*Hint:* Look at the proof of the Mean Value Property 3.3.) *(5 points)*

(b) Conclude from (a) the *Maximum Principle*: If the maximum of v can be found inside Ω, then v is constant. *(4 points)*

(c) Let now $u : \overline{\Omega} \to \mathbb{R}$ be a harmonic function. Show that:

(i) $\|\nabla u\|^2$ is subharmonic. *(3 points)*

(ii) If $f : \mathbb{R} \to \mathbb{R}$ is a smooth and convex function then $f \circ u$ is subharmonic. *(3 points)*