Introduction to Partial Differential Equations

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Exercise sheet 1

September 4th, 2018

(submit your solutions until September 10th, 4pm in A5, C, east entrance, post box: 46236)

1. Generalised Leibniz rule.

Let $u, v : \Omega \to \mathbb{R}$ be smooth enough functions on an open subset $\Omega \subset \mathbb{R}^n$.
Show for all multiindices $\gamma \in \mathbb{N}_0^n$ the following product rule:

$$\partial^\gamma(uv) = \sum_{0 \le \delta \le \gamma} \binom{\gamma}{\delta} \partial^\delta u \partial^{\gamma-\delta} v := \sum_{\delta_1=0}^{\gamma_1} \binom{\gamma_1}{\delta_1} \dots \sum_{\delta_n=0}^{\gamma_n} \binom{\gamma_n}{\delta_n} \partial^\delta u \partial^{\gamma-\delta} v.$$

(10 points)

2. Harmony.

Let $\Omega \subset \mathbb{R}^n$ be an open subset.

(a) Let $u, v : \Omega \to \mathbb{R}$ be harmonic functions.
Show that the function $w(x) := u(x) \cdot v(x)$ is harmonic if and only if ∇u and ∇v are perpendicular. *(5 points)*

(b) Let $f \in C^2(\Omega)$ and $g \in C^1(\Omega)$. Show that

$$g \triangle f = \operatorname{div}(g \nabla f) - \nabla f \cdot \nabla g.$$

(5 points)

(c) Let $u : \mathbb{R}^2 \to \mathbb{R}$ two times continiously differentiable.

(i) Show that $\triangle u$ has the following form in polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$:

$$\triangle u = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.$$

(10 points)

(ii) Show that the function $u(x, y) = r^n \cos n\theta$ is harmonic for any positive integer n.
(5 points)

3. A well structured algebra.

Let $f, g \in C_0^\infty(\mathbb{R}^n)$.
Show that the convolution defined by

$$(g * f)(x) := \int_{\mathbb{R}^n} g(x-y) f(y) d^n y = \int_{\mathbb{R}^n} g(y) f(x-y) d^n y$$

is commutative and associative.

(7 points)

4. In Color.

Let Ω be a region in $\mathbb{R}^n$ and N the outer unit normal vector field on $\partial\Omega$. Let u, v be two C^2 real-valued functions on $\overline{\Omega}$.

(a) Prove the 1st Green formula

$$\int_\Omega v \triangle u dx = -\int_\Omega \nabla u \cdot \nabla v dx + \int_{\partial\Omega} v \nabla u \cdot N d\sigma.$$

(*Hint:* Use the divergence theorem for $\vec{F} = v\nabla u$.) *(4 points)*

(b) Using the 1st Green formula, prove the 2nd Green formula

$$\int_\Omega (v \triangle u - u \triangle v) dx = \int_{\partial\Omega} (v \nabla u - u \nabla v) \cdot N d\sigma.$$

(4 points)