Introduction to Partial Differential Equations

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(submit your solutions until Wednesday, October 3rd, 4pm in A5, C, east entrance, post box: 46236)

11. Linear Partial Differential Equations.

(a) Let $b : \mathbb{R}^n \to \mathbb{R}^n$ and $c : \mathbb{R}^n \to \mathbb{R}$ continously differentiable functions. Then, let $x : I \to \mathbb{R}^n$ a solution of the ordinary differential equation

$$\dot{x}(s) = b(x(s))$$

and $u : \mathbb{R}^n \to \mathbb{R}$ be a solution of the homogenous, linear partial differential equation

$$b(x) \cdot \nabla u(x) + c(x)u(x) = 0.$$

Show, that the function $z(s) := u(x(s))$ is a solution of the ordinary differential equation

$$\dot{z}(s) = -c(x(s))z(s).$$

(8 points)

(b) Consider a PDE of the form $F(\nabla u(x), u(x), x) = 0$. Show that the to the method of characteristics corresponding curves $x(s)$ and $z(s) = u(x(s))$ can be described by ordinary differential equations without the dependence of $p(s) = (\nabla u)(x(s))$, if the function $F(p, z, x)$ is linear in p, that is, if F is of the form

$$F(p, z, x) = b(z, x) \cdot p + c(z, x)$$

with continously differentiable functions $b : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $c : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. *(6 points)*

(c) Use (b) to find again the solution of the inhomogenous transport equation. *(6 points)*

12. Solving PDE's Solve with the help of the method of characters the following boundary value problems. In each of the following exercises g is a continously differentiable function with matching domain.

(a) $x_1 \frac{\partial u}{\partial x_1} + x_2 \frac{\partial u}{\partial x_2} = 2u, \quad u(x_1, 1) = g(x_1), \quad$ for $x_1 \in \mathbb{R}, x_2 > 0$. *(6 points)*

(b) $x_1 \frac{\partial u}{\partial x_2} - x_2 \frac{\partial u}{\partial x_1} = u \quad u(x_1, 0) = g(x_1), \quad$ for $x_1, x_2 > 0$. *(6 points)*

(c) $x_1 \frac{\partial u}{\partial x_1} + 2x_2 \frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_3} = 3u, \quad u(x_1, x_2, 0) = g(x_1, x_2), \quad$ for $x_1, x_2 \in \mathbb{R}$. *(8 points)*

(d) $u \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = 1, \quad u(x_1, x_1) = \frac{1}{2}x_1, \quad$ for $x_1, x_2 > 2$. *(10 points)*
(*Hint:* It is not mandatory to make a coordinate transformation)