Chapter 1

Introduction

A partial differential equation is an equation on the partial derivatives of a function depending on at least two variabels.

Definition 1.1. *A possibly vector valued equation of the following form*

$$F\left(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x\right) = 0$$

is called partial differential equation of order k. Here F is a given function and u an unknown function. The expressions $D^k u$ denotes the vector of all partial derivatives of the function u of order k. The function u is called a solution of the differential equation, if u is k times differentiable and obeys the partial differential equation.

On subsets of $\mathbb{R}^n$ we denote the partial derivatives of higher order by $\partial^\gamma = \prod_i \partial_i^{\gamma_i} = \prod_i (\frac{\partial}{\partial x_i})^{\gamma_i}$ with multiindices $\gamma \in \mathbb{N}_0^n$ of length $|\gamma| = \sum_i \gamma_i$. The multiindices are ordered by $\delta \leq \gamma \iff \delta_i \leq \gamma_i$ for $i = 1, \ldots, n$. The partial derivatives only act on the direct subsequent functions. They only act on a product of functions if the product is included in brakets.

Exercise 1.2. *Show for all $\gamma \in \mathbb{N}_0^n$ the generalised Leibniz rule*

$$\partial^\gamma(uv) = \sum_{0\leq\delta\leq\gamma} \binom{\gamma}{\delta} \partial^\delta u \partial^{\gamma-\delta} v := \sum_{\delta_1=0}^{\gamma_1} \binom{\gamma_1}{\delta_1} \ldots \sum_{\delta_n=0}^{\gamma_n} \binom{\gamma_n}{\delta_n} \partial^\delta u \partial^{\gamma-\delta} v.$$

1.1 Examples

A. Linear differential equations

1. Laplace equation. $$\triangle u := \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = 0.$$

The solutions are called harmonic functions. The Laplace eqaation is a homogenous linear partial differential equation of second order.
The corresponding inhomogenous differential equation is called **Poisson equation.** $-\triangle u = f.$
Here f is a given function and u an unknown function.

2. Helmholtz equation. $-\triangle u - \lambda u = 0.$

Here $\lambda \in \mathbb{R}$ is a real parameter and u the unknown function. This is a simple example of the Poisson equation.

3. Linear transport equation. $\dot{u} + b \cdot \nabla u = 0.$

Here b is a vector field on $\Omega \subseteq \mathbb{R} \times \mathbb{R}^n$ and u the unknown function.

4. Liouville equation. $\dot{u} + \nabla(b \cdot u) = 0.$

Here b is a given $\mathbb{R}^n$-valued vector field on an open domain $\Omega \subseteq \mathbb{R} \times \mathbb{R}^n$ and u the unknown function on the same domain Ω. This linear differential equation of first order is similar to the transport equation.

5. Heat equation. $\dot{u} - \triangle u = 0.$

6. Schrödinger equation. $\imath \dot{u} + \triangle u = 0.$

Here u is an unknown complex valued function. The Schröding equation differs from the heat equation by a factor $\imath$, which results in very different behaviors of the corresponding solutions.

7. Kolmogorov equation. $$\dot{u} - \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} = 0.$$

This is a generalised version of the heat equation.

8. Fokker-Planck equation. $$\dot{u} - \sum_{i,j=1}^{n} \frac{\partial^2 a_{ij}(t,x) u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \frac{\partial b_i(t,x) u}{\partial x_i} = 0.$$

9. Wave equation. $$\frac{\partial^2 u}{\partial t^2} - \triangle u = 0.$$

10. General wave equation. $$\frac{\partial^2 u}{\partial t^2} - \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t,x) \frac{\partial u}{\partial x_i} = 0.$$

This equation generalises the ordinary wave equation in the same way as the Kolmogorov equation generalises the heat equation.

11. Airy differential equation. $\dot{u} + \frac{\partial^3 u}{\partial x^3} = 0.$

Here u is an unknown function on a domain $\Omega \subseteq \mathbb{R} \times \mathbb{R}$.

12. Beam equation. $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^4 u}{\partial x^4} = 0.$

B. Nonlinear differential equations

1. Eikonal equation. $|\nabla u| = 1.$

2. nonlinear Poisson equation. $-\triangle u = f(u).$

Here $f : \mathbb{R} \to \mathbb{R}$ is a given function and u the unknown function on $\Omega \subseteq \mathbb{R}^n$.

3. Minimal surface equation. $\nabla \cdot \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0.$

The graphs of solutions of the minimal surface equations are minimal surfaces. The area of such hypersurfaces in $\mathbb{R}^{n+1}$ do not change in first order with respect to infinitesimal deformations. Soap bubbles are examples.

4. Monge-Ampere equation. $\det\left(\nabla\nabla^t u\right) = f.$

Here f is a given function on a domain $\Omega \subseteq \mathbb{R}^n$ and u is the unknown function. The leftt hand side is the determinant of the Hessian of u.

5. Hamilton-Jacobi equation. $\dot{u} + H(\nabla u, x) = 0.$

Here H is a given Hamilton function on a subset of $\mathbb{R}^n \times \mathbb{R}^n$ and u is the unknown functions on the corresponding domain in $\mathbb{R}^n$.

6. Scalar conservation law. $\dot{u} + \nabla \cdot F(u) = 0.$

Here F is a given $\mathbb{R}^n$-valued functon and u is the unknown function on a domain $\Omega \subseteq \mathbb{R} \times \mathbb{R}^n$. This differential equation implies that the time derivative of the integral of u over a given domain in $\mathbb{R}^n$ is equal to the integral of $F(u)$ over the boundary of the domain. Therefore $F(u)$ descibes the flux density of the conserved quantity.

7. Burgers equation. $\dot{u} + u\frac{\partial u}{\partial x} = 0.$

Here u is the unknown function on a domain $\Omega \subseteq \mathbb{R} \times \mathbb{R}$. This is an example of an conservation law with flux density $F(u) = u^2/2$.

8. Reaction-diffusion equation. $\dot{u} - \triangle u = f(u).$

Here $f : \mathbb{R} \to \mathbb{R}$ is a given function and u is the unknown function.

9. Porous medium equation. $\dot{u} - \triangle (u^\gamma) = 0.$

Here $\gamma \geq 1$ is a given exponent. This equation describes the propagation of an ideal gas in a porous medium.

10. Nonlinear wave equation. $\frac{\partial^2 u}{\partial t^2} - \triangle u = f(u).$

Here $f : \mathbb{R} \to \mathbb{R}$ is a given and u is the unknown function.

11. Korteweg-de-Vries equation. $4\dot{u} - 6u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} = 0.$

There exists a Lax representation for this equation:

$$\dot{L} = [A, L] \qquad \text{mit} \qquad L := \frac{\partial^2}{\partial x^2} + u \qquad A := \frac{\partial^3}{\partial x^3} + \frac{3u}{2}\frac{\partial}{\partial x} + \frac{3}{4}\frac{\partial u}{\partial x}.$$

This representation was the starting point of a new approach to integrable systems.

C. Linear systems of differential equations.

1. Linear elasticity. $\mu \triangle u + (\lambda + \mu)\nabla(\nabla \cdot u) = 0.$

Here $\lambda > 0$ and $\mu > 0$ are constants and u the unknown function.

2. Elastic wave equation. $\frac{\partial^2 u}{\partial t^2} - \mu \triangle u - (\lambda + \mu)\nabla(\nabla \cdot u) = 0.$

3. Maxwell equation.

$$\begin{aligned} \dot{E} - \nabla \times B &= -4\pi j & \dot{B} + \nabla \times E &= 0 \\ \nabla \cdot E &= 4\pi\rho & \nabla \cdot B &= 0. \end{aligned}$$

Here the real valued charge density ρ and the $\mathbb{R}^3$-valued current density j are given functions on space time $\mathbb{R} \times \mathbb{R}^3$ and the electric field E and the magnetic field B are

$\mathbb{R}^3$-valued unknown functions. The electric and current densities obey the conservation law

$$\dot{\rho} + \nabla \cdot j = 0.$$

4. Cauchy-Riemann equation. $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Here $(u, v) : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (u, v)$ are real and imaginary parts of a holomorphic functions on a domain in the complex plane $x + \imath y = z \in \mathbb{C}$.

D. Nonlinear systems of differential equations.

1. Euler equation. $\dot{u} + u \cdot \nabla u + \nabla p = 0 \qquad \nabla \cdot u = 0.$

Here $u : \mathbb{R}^3 \to \mathbb{R}^3$ is the velocity field of an incompressible fluid without viscosity with pressure p.

2. Navier-Stokes equation. $\dot{u} + u \cdot \nabla u - \triangle u + \nabla p = 0 \qquad \nabla \cdot u = 0.$

Here $u : \mathbb{R}^3 \to \mathbb{R}^3$ is the velocity field of an incompressible viscous fluid with pressure p.

3. Einstein field equations. $$R_{ij} - \frac{1}{2} g_{ij} R = \kappa T_{ij}.$$

Here T_{ij} is stress energy tensor of a given distribution of mass on the space time and g_{ij} is the unknown metric on space time with signature $(1, 3)$. R_{ij} is the corresponding Ricci tensor and R scalar curvature.

$$\Gamma_{ij}^{k} := \frac{1}{2} \sum_{l=0}^{3} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right) \qquad \left(g^{ij}\right) := \left(g_{ij}\right)^{-1} \text{ inverse Metrik}$$

$$R_{ij} := \sum_{k=0}^{3} g^{kl} \left(\frac{\partial \Gamma_{ij}^{k}}{\partial x^k} - \frac{\partial \Gamma_{ik}^{k}}{\partial x^j} + \sum_{l=0}^{3} \left(\Gamma_{lk}^{k} \Gamma_{ij}^{l} - \Gamma_{lj}^{k} \Gamma_{ik}^{l} \right) \right) \qquad R := \sum_{i,j=0}^{3} g^{ij} R_{ij}.$$

4. Ricci flow. $\dot{g}_{ij} = -2 R_{ij}.$

This differential equation descibes a diffusion flow on the metric of a Riemannian manifold. It equalizes inhomogenities and anisotropies of the metric and converges in the long time limit to a metric with large isometry groups. In the 1970ties Richard Hamilton set up a strategy to prove the geometrization conjecture of Thurston with the Ricci flow. This conjecture claims that every compact 3-manifold may be decomposed in pieces with transitive isometry groups. Hamilton tried to control the long time behaviour of the Ricci flow on compact 3-manifolds. The russian mathematician Grisha Perelman published in 2003 3 preprimts in the net, which overcame the last impasses of this program. This was a great success of geometric analysis.

1.2 Divergence Theorem

Definition 1.3. *(partition uf unity) For a given family $(U_\alpha)_{\alpha\in A}$ of open subsets of $\mathbb{R}^n$ with union $\bigcup_{\alpha\in A} U_\alpha = \Omega \subset \mathbb{R}^n$ a smooth partition of unity is a countable family $(h_l)_{l\in\mathbb{N}}$ of smooth functions $h_l : \Omega \to [0,1]$ with the following properties:*

(i) *Each $x \in \Omega$ has a neighbourhood where all but finitely h_l vanish identically.*

(ii) *For all $x \in \Omega$ we have $\sum_{l=1}^{\infty} h_l(x) = 1$.*

(iii) *Each h_l vanishes outside a compact subset of U_α for some $\alpha \in A$.*

For every family of open subsets of $\mathbb{R}^n$ there exists a smooth partititon of unity.

Definition 1.4. *For each $n\times(n-1)$-matrix A there exists a unique row vector $A^\# \in \mathbb{R}^n$, such that $\det(A,x) = x^t \cdot A^\#$ holds for all $x \in \mathbb{R}^n$. This vector is orthogonal to the image of A. The length of this vector is the aerea of the image of $[0,1]^{n-1}$ in $\mathbb{R}^n$ with respect to A. For a $n \times n$-matrix A we have $(A|_{\mathbb{R}^{n-1}})^\# = \det(A)(A^{-1})^t e_n$.*

Definition 1.5. *The boundary of an open susbet $\Omega \subseteq \mathbb{R}^n$ is called continuously differentiable, if the closure $\bar{\Omega}$ is covered by open subsets $O \subseteq \mathbb{R}^n$ with continuously differentiabel maps $\Phi : U \to O$ with continuously differentialble inverse maps $\Phi^{-1} : O \to U$, which map $O \cap \Omega$ into the upper half plane $\{x \in \mathbb{R}^n \mid x_n > 0\}$ and $O \cap \partial\Omega$ into $\mathbb{R}^{n-1} = \{x \in \mathbb{R}^n \mid x_n = 0\}$. For $\det \Phi' \gtrless 0$ the integral of a function f on $\partial\Omega$ is defined in terms of a partition of unity $(h_l)_{l\in\mathbb{N}}$ as*

$$\int_{\partial\Omega} f \, \mathrm{d}\sigma = \sum_{l\in\mathbb{N}} \int_{U\cap\mathbb{R}^{n-1}} (h_l f) \circ \Phi \left| (\Phi'|_{\mathbb{R}^{n-1}})^\# \right| \mathrm{d}\mu_{\mathbb{R}^{n-1}} \text{ with } h_l|_{\mathbb{R}^n\setminus O} = 0.$$

For a $\mathbb{R}^n$-valued function f and the outer normal N of $\partial\Omega$ we define

$$\int_{\partial\Omega} f \cdot N \, \mathrm{d}\sigma := \mp \sum_{l\in\mathbb{N}} \int_{U\cap\mathbb{R}^{n-1}} ((h_l f) \circ \Phi) \cdot (\Phi'|_{\mathbb{R}^{n-1}})^\# \, \mathrm{d}\mu_{\mathbb{R}^{n-1}} \text{ with } h_l|_{\mathbb{R}^n\setminus O} = 0.$$

The sign on the right hand side is $-$ for $\det \Phi' > 0$ and $+$ for $\det \Phi' < 0$.

For the calculation of these integrals it suffices to know the maps Φ on $U \cap \mathbb{R}^{n-1}$. Continuouslt differentiable embeddings $\Psi : U \cap \mathbb{R}^{n-1} \to O \cap \partial\Omega$, whose derivatives Ψ' have rank $n-1$ have continuosly differentiable extensions to small neighbourhoods of $U \cap \mathbb{R}^{n-1}$ with continuously differentiable inverse mappings. Therefore it suffices to assume the existence of such mappings $\Psi = \Phi|_{U\cap\mathbb{R}^{n-1}}$.

Lemma 1.6. *On the $d \times d$-matrices the* det *defines a differentiable map with derivative*

$$\frac{d}{dt}\det(A+tB)\bigg|_{t=0} = \operatorname{trace}(\det(A)A^{-1}B).$$

Proof. For two $d \times d$-matrices A and B, whose firts is invertible we have

$$\det(A+tB) = \det(A)\det(\mathbb{1}+tA^{-1}B) = t^d\det(A)\det(t^{-1}\mathbb{1}+A^{-1}B) \text{ f"ur } t \neq 0.$$

At $t=0$ the derivative with respect to t is equal to $\det(A)$ times the second highest coefficient of the characteristic polynomial of $-A^{-1}B$ which is equal to $\det(A)\operatorname{trace}(A^{-1}B)$. **q.e.d.**

Theorem 1.7. *(Divergence Theorem) Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded subset with two times differentiable boundary. Let $f : \bar{\Omega} \to \mathbb{R}^n$ continuous and differentiable on Ω such that all first partial derivatives have continuous extensions to $\bar{\Omega}$. Then we have*

$$\int_\Omega \nabla \cdot f \,\mathrm{d}\mu = \int_{\partial\Omega} f \cdot N \,\mathrm{d}\sigma$$

Here N is the outer normal and $N\,\mathrm{d}\sigma$ the correspinding measure on the boundary $\partial\Omega$.

Proof. By using a smooth partititon of unity it suffices to prove the stetament for a function f which vanishes outside a closed subset of one of the open subsets O in Definition 1.5. We apply Lemma 1.6 and calculate for $\tilde{f} := \det(\Phi')(\Phi')^{-1}(f\circ\Phi)$

$$\begin{aligned}\nabla\cdot\tilde{f} &= \det(\Phi')\sum_{ijkl}(\Phi')^{-1}_{ij}\frac{\partial^2\Phi_j}{\partial x_k\partial x_i}(\Phi)^{-1}_{kl}f_l\circ\Phi - \det(\Phi')\sum_{ijkl}(\Phi')^{-1}_{ij}\frac{\partial^2\Phi_j}{\partial x_i\partial x_k}(\Phi')^{-1}_{kl}f_l\circ\Phi \\ &\quad + \det(\Phi')\operatorname{trace}((\Phi')^{-1}(f'\circ\Phi)\Phi') = \det(\Phi')\operatorname{trace}(f'\circ\Phi) = \det(\Phi')(\nabla\cdot f)\circ\Phi.\end{aligned}$$

With Jacobi's transformation formula we obtain $\int_O \nabla f\,\mathrm{d}\mu = \int_U \nabla\tilde{f}\,\mathrm{d}\mu$. Hence it suffices to show

$$\begin{aligned}\int_U \nabla\tilde{f}\,\mathrm{d}\mu &= -\int_{U\cap\mathbb{R}^{n-1}}(f\circ\Phi)\cdot(\Phi'|_{\mathbb{R}^{n-1}})^{\#}\,\mathrm{d}\mu_{\mathbb{R}^{n-1}} \\ &= -\int_{U\cap\mathbb{R}^{n-1}}\det(\Phi')^{-1}(\Phi'\tilde{f})\cdot\det(\Phi')((\Phi')^{-1})^t e_n\,\mathrm{d}\mu_{\mathbb{R}^{n-1}} = -\int_{U\cap\mathbb{R}^{n-1}}\tilde{f}_n\,\mathrm{d}\mu_{\mathbb{R}^{n-1}}.\end{aligned}$$

We extend the function $\tilde{f}$ continuously differentiable to a quader with one face inside of the the hyperplane $\mathbb{R}^{n-1} \subseteq \mathbb{R}^n$. Since $\tilde{f}$ vanishes on all faces of this quader with the exception of the unique face inside $\mathbb{R}^{n-1}$ we calculate the integral on the left hand side and obtain the integral on the right hand side. **q.e.d.**

1.3 Existence of solutions

In order to demonstrate the difference between ordinary and partial differential equations we shall present an example of a partial differential equation with smooth coefficients without solutions. This example is a simplification by Nirenbereg of an example of H. Levy.

For a given complex-valued function f on a domain $(x, y) \in \mathbb{R}^2$ we look for a complex valued solution u on the same domain of the following differential equations:

$$\frac{\partial u}{\partial x} + \imath x \frac{\partial u}{\partial y} = f(x, y).$$

We impose the following two conditions on the smooth function f:

(i) $f(-x, y) = f(x, y)$

(ii) there exists a sequence of positive numbers $\varrho_n \downarrow 0$ converging to zero, such that f vanishes on a nieghbourhood of the circles $\partial B(0, \varrho_n)$ in contrast to non-vanishing integrals $\int_{B(0,\varrho_n)} f(x, y)\, dx\, dy \neq 0$.

If $h : \mathbb{R} \to [0, \infty)$ is a smooth periodic function vanishing on an intervall but not on $\mathbb{R}$, then $f(x) := \exp(-1/|x|)h(1/|x|)$ has these two properties.

Now we shall prove by contradiction that there exists no continuously differentiable solution u in a neighbourhood of $(0, 0) \in \mathbb{R}^2$.

step 1: If the function $u(x, y)$ is a solution, then due to (i) $-u(-x, y)$ is also a solution. Hence we may replace $u(x, y)$ by $\frac{1}{2}(u(x, y) - u(-x, y))$ and assume $u(-x, y) = -u(x, y)$.

step 2: We claim that every solution u vanishes on the circles $\partial B(0, \varrho_n)$. In fact, we transform small annuli A onto domains $\tilde{A}$ in $\mathbb{R}^2$:

$$A \to \tilde{A}, \qquad (x, y) \mapsto \begin{cases} (x^2/2, y) & \text{for } x \geq 0 \\ (-x^2/2, y) & \text{for } x < 0. \end{cases}$$

These transformations are homoemorphisms from A onto $\tilde{A}$. On the sub domains $\tilde{A}_+ = \left\{(s, y) \in \tilde{A} \mid s > 0\right\}$ the function $\tilde{u}(s, y) = u(x^2/2, y)$ is holomorphic:

$$2\bar{\partial}\tilde{u} = \frac{\partial \tilde{u}(s, y)}{\partial s} + \imath \frac{\tilde{u}(s, y)}{\partial y} = \frac{dx}{ds}\frac{\partial u(x, y)}{\partial x} + \imath \frac{\partial u(x, y)}{\partial y} = \frac{1}{x}\left(\frac{\partial u(x, y)}{\partial x} + \imath x \frac{\partial u(x, y)}{\partial y}\right) = 0.$$

Due to step 1. the function $\tilde{u}$ vanises on the line $s = 0$. This implies that $\tilde{u}$ together with the Taylor series vanishes identically on $\tilde{A}_+$ and due to step 1 on $\tilde{A}$.
step 3: The divergence theorem yeilds a contradiction to the assumption (ii):

$$\begin{aligned}\int_{B(0,\varrho_n)} f \,\mathrm{d}x\,\mathrm{d}y &= \int_{B(0,\varrho_n)} \left(\frac{\partial u}{\partial x} + \imath x \frac{\partial u}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \int_{B(0,\varrho_n)} \nabla \cdot \begin{pmatrix} u \\ \imath x u \end{pmatrix} \mathrm{d}x\,\mathrm{d}y \\ &= \int_{\partial B(0,\varrho_n)} \begin{pmatrix} u \\ \imath x u \end{pmatrix} \cdot N(x,y)\,\mathrm{d}\sigma(x,y) = 0,\end{aligned}$$

Therefore the given differential equation has no continuously differentiable solution.

This example also implies that the following partial differential equation with smooth real coefficients has no four times differentiable solution:

$$\left(\frac{\partial}{\partial x} + \imath x \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \imath x \frac{\partial}{\partial y}\right)^2\left(\frac{\partial}{\partial x} + \imath x \frac{\partial}{\partial y}\right) u = \left(\left(\frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2}\right)^2 + \frac{\partial^2}{\partial y^2}\right) u = f.$$

1.4 Distributions

Our investigation of partial differential equations aims to find as many solutions as possible and in addition conditions which uniquely determines the solutions. The solutions and theire existence depend on the notion of a solution. Clearly all partial derivatives of a solution which occure in the partial differential equation have to exist. We might use several possible generalisations of derivatives in order to define such solutions. In this section we introduce generalised functions which can be differentiated infinitely many times. For this achievement we have to pay a price: these generalised functions cannot be multiplied with each other. Linear partial differential equations extend to well defined equations on such generaliseed functions. We call generalised functions solving the linear partial differential equations weak solutions or solutions in the sense of distributions. There exist other notions of weak solutions which also apply to non-linear partial differential equations. An example of more general functions with finitely many derivaitves are so called sobolev spaces. These sobolev spaces are introduced in more advanced lectures on partial differential equations. The elements of the sobolev spaces are distributions. So the distributions which we introduce now are the most general functions with derivatives.

The support $\operatorname{supp} f$ of a function f is the closure of $\{x \mid f(x) \neq 0\}$. On an open set $\Omega \subseteq \mathbb{R}^n$ let $C_0^\infty(\Omega)$ denote the algebra of smooth functions with compact support in Ω, i.e. $\operatorname{supp} f \Subset \Omega$. Each $f \in L^1(\Omega)$ defines a linear map

$$F : C_0^\infty(\Omega) \to \mathbb{R}, \qquad\qquad \phi \mapsto \int_\Omega f\phi\,\mathrm{d}\mu.$$

Generalised funtions on Ω are such linear fomrs F on $C_0^\infty(\Omega)$. We call the elements of $C_0^\infty(\Omega)$ in the domain of the linear form F test functions. By integration by parts we obtain

$$\int_\Omega \partial_i f \phi \, \mathrm{d^n}x = -\int_\Omega f \partial_i \phi \, \mathrm{d^n}x.$$

Therefore such generalised functions have infinitley many derivatives. For any linear form F on $C_0^\infty(\Omega)$ we define the partial derivatives as

$$\partial_i F : C_0^\infty(\Omega) \to \mathbb{R}, \qquad \phi \mapsto -F(\partial_i \phi).$$

The vector space of test functions is infinite dimensional. In order to avoid abstract non sense we should impose some continuity on the linear forms F. The derivative of a continuous functional F is again continuous, if the derivatives are linear continuous maps on the space $C_0^\infty(\Omega)$. For $f \in L^1(\Omega))$ the corresponding linear functionals F are continous with respect to the supremum norm on kompact subsets of Ω. We define for any compact subset $K \subset \Omega$ and every multi index α the following semi norm:

$$\|\cdot\|_{K,\alpha} : C_0^\infty(\Omega) \to \mathbb{R}, \qquad \phi \mapsto \|\phi\|_{K,\alpha} := \sup_{x\in K} |\partial^\alpha \phi(x)|\,.$$

Definition 1.8. *On an open subset $\Omega \subseteq \mathbb{R}^n$ the space of distributions $\mathcal{D}'(\Omega)$ is defined as the vector space space of all linear maps $F : C_0^\infty(\Omega) \to \mathbb{R}$ which are continuous with respect to the semi norms $\|\cdot\|_{K,\alpha}$; i.e. for each compact $K \subset \Omega$ there exist finitley many multi indices $\alpha_1, \ldots, \alpha_M$ and constants $C_1 > 0, \ldots, C_M > 0$ such that the following inequality holds for all test functions $\phi \in C_0^\infty(\Omega)$ with compact support in K:*

$$|F(\phi)| \le C_1 \|\phi\|_{K,\alpha_1} + \ldots + C_M \|\phi\|_{K,\alpha_M}.$$

The support $\operatorname{supp} F$ of a distribution $F \in \mathcal{D}'(\Omega)$ is defined as the complement of the union of all open subsets $O \subset \Omega$, such that F vanishes on all test funcitons ϕ whose support is contained in O. We denote the euclidean length of $x \in \mathbb{R}^n$ by $|x|$. The testfunction

$$\phi(x) := \begin{cases} \exp\left(\frac{1}{|x|^2-1}\right) & \text{for } |x| < 1 \\ 0 & \text{for } |x| \ge 1 \end{cases}$$

has support $\overline{B(0,1)}$ and is non-negative. By rescaling of x and ϕ and by translations we obtain for each ball $B(x_0, \epsilon)$ a unique non-negative test function $\phi_{B(x_0,\epsilon)}$ with $\operatorname{supp} \phi_{B(x_0,\epsilon)} = \overline{B(x_0,\epsilon)}$ with $\int \phi_{B(x_0,\epsilon)} \, \mathrm{d}\mu = 1$. In particular, there exists for every open subset $O \subset \Omega$ a non-negative test function with support contained O. Every continuous function f on Ω which does no vanish identically takes values in $(-\infty, \epsilon) \cup (\epsilon, \infty)$

for some $\epsilon > 0$ on some properly chosen open ball. Therefore there exists $\phi \in C_0^\infty(\Omega)$ with $\int_\Omega f\phi \,\mathrm{d}\mu \neq 0$. The following distribution does not correspond to a usual function:

$$\delta : C_0^\infty(\Omega) \to \mathbb{R} \qquad\qquad \phi \mapsto \phi(0).$$

A corresponding function would vanish on $\mathbb{R}^n \setminus \{0\}$ and would have a total integral one. Since $\{0\}$ has maesure zero such a function does not exist. This generalised function is called Dirac's δ-function. The familiy of distributions, which corresponds to the functions $\phi_{B(0,\epsilon)}$ converge in the limit $\epsilon \downarrow 0$ to this dirtribution. The support of all derivatives of this distribution contains only the point $0 \in \Omega$.

The product of a distribution with a function $g \in C^\infty(\Omega)$ is defined as

$$gF : C_0^\infty(\Omega) \to \mathbb{R}, \qquad\qquad \phi \mapsto F(g\phi).$$

This product makes the embedding $C^\infty(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ of the space of smooth functions into the space a distributions to a homomorphism of modules over the algebra $C^\infty(\Omega)$. However, even the product of a distribution with a continuous non-smooth functions is not defined. The convolution defines another product on $C_0^\infty(\mathbb{R}^n)$:

$$(g * f)(x) := \int_{\mathbb{R}^n} g(x-y)f(y)\,\mathrm{d^n}y = \int_{\mathbb{R}^n} g(y)f(x-y)\,\mathrm{d^n}y.$$

This product is commutative and associative (Exercise). In order to extend this product to a product between a smooth functions and a distributions we calculate:

$$\begin{gathered}\int\limits_{\mathbb{R}^n} \phi(g*f)\,\mathrm{d^n}x = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \phi(x)g(x-y)f(y)\,\mathrm{d^n}y\,\mathrm{d^n}x = \int\limits_{\mathbb{R}^n} \phi(x)\int\limits_{\mathbb{R}^n} (\mathsf{T}_x\mathsf{P}g)(y)f(y)\,\mathrm{d^n}y\,\mathrm{d^n}x \\ = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi(x)g(x-y)f(y)\,\mathrm{d^n}x\,\mathrm{d^n}y = \int_{\mathbb{R}^n} (\phi * \mathsf{P}g)f\,\mathrm{d^n}y \\ \text{with } \mathsf{T}_x : C_0^\infty(\Omega) \to C_0^\infty(x+\Omega), \quad \phi \mapsto \mathsf{T}_x\phi, \text{ and } (\mathsf{T}_x\phi)(y) = \phi(y-x) \\ \text{and } \mathsf{P} : C_0^\infty(\Omega) \to C_0^\infty(-\Omega), \quad \phi \mapsto \mathsf{P}\phi, \text{ with } (\mathsf{P}\phi)(y) = \phi(-y).\end{gathered}$$

Therefore we define for $g \in C_0^\infty(\mathbb{R}^n)$ and $F \in \mathcal{D}'(\mathbb{R}^n)$

$$g * F : \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto F(\mathsf{T}_x\mathsf{P}g) \text{ or equivalently } g * F : C_0^\infty(\mathbb{R}^n) \to \mathbb{R}, \quad \phi \mapsto F(\phi * \mathsf{P}g).$$

Lemma 1.9. *The convolutions of a distributions with a test functions $g \in C_0^\infty(\Omega)$ is a distribution which corresponds to a smooth function. The support of this distribution is cotained in the point-wise sum of the supports of the functions and the distribution.*

Proof. For each $F \in \mathcal{D}'(\Omega)$ the linearity and continuity imply

$$g * F(\phi) = F(\mathsf{P}g * \phi) = \int_{\mathbb{R}^n} F(\mathsf{T}_x \mathsf{P} g)\phi(x)\,\mathrm{d}^{\mathrm{n}}x.$$

Due to the continuity of F with respect to the semi norms $\|\cdot\|_{K,0}$ the functions $x \mapsto F(\mathsf{T}_x\mathsf{P}g)$ is continuous. Furthermor, these functions are smooth since $\frac{\mathsf{T}(y+\epsilon h)-\mathsf{T}(y)}{\epsilon}$ converges in the limit $\epsilon \to 0$ on the space $C^\infty(\Omega)$ with respect to the topology induced by the semi norms $\|\cdot\|_{K,\alpha}$ to the operator $\mathsf{T}(y)\frac{\partial}{\partial x_i}$.

If $x \mapsto F(T_x P g)$ does not vanish on a neighbourhood of a point x, then $g(x-y) \neq 0$ for an element $y \in \operatorname{supp} F$. Hence $x = y + (x-y)$ is the sum of an element of $\operatorname{supp} F$ with an element of $\operatorname{supp} g$. **q.e.d.**

This Lemma implies that even the convolution of a distribution $F \in \mathcal{D}'(\mathbb{R}^n)$ with a distribution $G \in \mathcal{D}'(\mathbb{R}^n)$ with compact support $\operatorname{supp} G$ is a well defined distribution:

$$F * G : C_0^\infty(\Omega) \to \mathbb{R}, \qquad \phi \mapsto F(\phi * \mathsf{P}G) \text{ mit} \qquad \mathsf{P}G(\phi) := G(\mathsf{P}\phi).$$

In particular, the δ-distribution is the neutral element of the product defined by the convolution, i.e. the convolution with the δ-distribution maps each distribution onto itself. We introduced a familly of test functions $(\phi_{B(0,\epsilon)})_{\epsilon>0}$ which converge in the limit $\epsilon \downarrow 0$ to the δ-distribution. For each distribution $F \in \mathcal{D}'(\Omega)$ the family of smooth functions $f_\epsilon := \phi_{B(0,\epsilon)} * F$ converge in the limit $\epsilon \downarrow 0$ in a specific sense to the distribution F. Such a family $(\lambda_\epsilon)_{\epsilon>0}$ in $C_0(\mathbb{R}^n)$ with

$$\lambda_\epsilon \geq 0 \qquad \operatorname{supp}\lambda_\epsilon \subset \overline{B(0,\epsilon)} \qquad \int_{\mathbb{R}^n} \lambda_\epsilon\,\mathrm{d}^{\mathrm{n}}x = 1,$$

which converges in the limit $\epsilon \downarrow 0$ to the δ-distribution, is called mollifier. Now we can show that all distributions can be approximated by smooth functions.

Lemma 1.10. *Let $f \in C(\Omega)$ and $(\lambda_\epsilon)_{\epsilon>0}$ be a mollifier. In the limit $\epsilon \downarrow 0$ the family of smooth functions $\lambda_\epsilon * f$ converges uniformly on compact subets of Ω to f. For smooth functions the same holds for all derivatives of f.*

Proof. On compact subsets of Ω the continuous functions f are uniformly continuous. Each element x of the open set Ω is contained in an open ball $B(x,\epsilon) \subset \Omega$. For sufficiemtly small ϵ we have

$$|(\lambda_\epsilon * f)(x) - f(x)| = \left|\int_{B(x,\epsilon)} \lambda_\epsilon(x-y)(f(y)-f(x))\,\mathrm{d}^{\mathrm{n}}y\right| \leq \sup_{y \in B(x,\epsilon)} |f(y)-f(x)|.$$

This shows the uniform convergency $\lim_{\epsilon\downarrow 0}\lambda_\epsilon * f = f$. By definition of the convolution two smooth functions f and g obey

$$\frac{\partial(f*g)}{\partial x_i} = f * \frac{\partial g}{\partial x_i} = \frac{\partial f}{\partial x_i} * g.$$

Hence for $f \in C^\infty(\Omega)$ these arguments carry over to all partial derivatives of f. **q.e.d.**

Each element $f \in L^1_{\mathrm{loc}}(\Omega)$ defines in a canonical way a distribution

$$F_f : C_0^\infty(\Omega) \to \mathbb{R}, \qquad \phi \mapsto \int_\Omega f\phi \, \mathrm{d}\mu.$$

For $\phi \in C_0^\infty(\Omega)$ with support in a compact subset $K \subset \Omega$ and $f \in L^1(\Omega)$ gwe have

$$|F_f(\phi)| \leq \sup_{x\in K} |\phi(x)| \|f\|_{L^1(\Omega)}.$$

For $f \in L^1_{\mathrm{loc}}(\Omega)$ every compact subset $K \subset \Omega$ has a covering by open subsets $O_1, \ldots, O_L$ of Ω such that $f|_{O_l} \in L^1(O_l)$ for $l = 1, \ldots, L$. This shows $F_f \in \mathcal{D}'(\Omega)$:

$$|F_f(\phi)| \leq \sup_{x\in K} |\phi(x)| \sum_{l=1}^{L} \|f|_{O_l}\|_{L^1(O_l)} \quad \text{for} \quad \operatorname{supp}\phi \subset K.$$

Lemma 1.11. *(fundamental lemma of the calculus of variantions) If $f \in L^1_{\mathrm{loc}}(\Omega)$ obeys $F_f(\phi) \geq 0$ for all non-negative test functions $\phi \in C_0^\infty(\Omega)$, then f is non-negative almost everywhere. In particular the map $L^1_{\mathrm{loc}}(\Omega) \to \mathcal{D}'(\Omega)$, $f \mapsto F_f$ is injective.*

Proof. It suffices to prove the local statement for $f \in L^1(\Omega)$. We extend f to $\mathbb{R}^n$ by setting f on $\mathbb{R}^n \setminus \Omega$ equal to zero. The extended function is also denoted by f and belongs to $f \in L^1(\mathbb{R}^n)$. For a mollifier $(\lambda_\epsilon)_{\epsilon>0}$ we have

$$\begin{aligned}\|\lambda_\epsilon * f - f\|_1 &= \int_{\mathbb{R}^n} \left| \int_{B(0,\epsilon)} \lambda_\epsilon(y) f(x-y)\,\mathrm{d^n}y - f(x) \right| \mathrm{d^n}x \leq \\ &\leq \int_{B(0,\epsilon)} \int_{\mathbb{R}^n} \lambda_\epsilon(y)|f(x-y) - f(x)|\,\mathrm{d^n}x\,\mathrm{d^n}y \leq \sup_{y\in B(0,\epsilon)} \|f(\cdot - y) - f\|_1.\end{aligned}$$

If f is the chracteristic functions of a rectangle, then the supremum on the right hand side converges to zero for $\epsilon \downarrow 0$. Due to the triangle inequality the same holds for step fuctions, i.e. finite linear combinations of such functions. Since step functions are dense in $L^1(\mathbb{R}^n)$ for each $f \in L^1(\mathbb{R}^n)$ this supremum becomes abitrary small for sufficiently small ϵ. Hence the family of functions $(\lambda_\epsilon * f)_{\epsilon>0}$ converges in $L^1(\mathbb{R}^n)$ in the limit $\epsilon \downarrow 0$ to

f. Hence there exists a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ which converges to zero, with $\|f_{n+1} - f_n\|_1 \leq 2^{-n}$ for all $n \in \mathbb{N}$ and $f_n = \lambda_{\epsilon_n} * f$. This ensures that the series $|f_1| + \sum_{n\in\mathbb{N}} |f_{n+1} - f_n|$ converges in $L^1(\mathbb{R}^n)$. Furthermore, due to Lebesgues bounded convergency the seqeunce $(f_n)_{n\in\mathbb{N}}$ converges almost everywhere to f. The non-negativity of the mollifiers together with the assumption on F_f implies $(\lambda_\epsilon * f)(x) = F_f(\lambda_\epsilon(x - \cdot) \geq 0$. This indeed shows that f is almost everywhere non-negative.

In particular, if f belongs to the kernel of $f \mapsto F_f$, then f is almost everywhere non-negative and non-positive. So f vanishes almost everywhere. **q.e.d.**

A short and lucid introduction into the theory of distributions is contained in the first chapter of the book of Lars H"ormander: "Linear Partial Differential Operators".

1.5 Regularity of solutions

The regularity of a solution of a differential equation collects the local properties of the corresponding functions. The most general functions we shall consider are distributions with the lowest regularity. They contain the measurable functions with the next higher regularity. The elements of L^p_{loc} describe smaller families of functions, whose regularity increase with $p \in [1, \infty]$. The next smaller class are sobolev functions whose k-th order partial derivatives belong to L^p_{loc}. The regularity further increases for the functions in C^k. Finally we end with the smooth functions and the analytic functions with the highest regularity.

1.6 Boundary value problems

Our investigations of solutions of partiel differential equations aims for a complete characterisations of all solutions. In general partial differetial equaitons have an infinite dimensional space of solutions. A solution of an ordinary differential equations of m-th order is in many cases uniquely determined by fixing the values of the first m derivatives at some initial value of the parameter. For partial differential equations we want to find some similar characterisation. The soltions are functions on higher dimensional domains $\Omega \subset \mathbb{R}^n$. A natural generalisation of this conditions is the specification of the values of the solution and some of its derivatives on the boundary of the domain. The search for solutions which obey this futher specification are called boundary value problems. So one important aim in the investigation of partial differetial equations is to find boudary value problems, which have unique soltions. Furthermore, if we can also determine all possible boundary values, then the space of solutions is completely parameterised.