Chapter 3

Laplace Equation

One of the most important PDE's is the Laplace eqeation

$$\triangle u = \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = 0.$$

The corresponding inhomogenous PDE is Poisson's equation

$$-\triangle u = f.$$

Both equations are linear PDE's of second order with the unknown function $u : \mathbb{R}^n \to \mathbb{R}$. They show up in many situations. In physics they descibe for example the potential of an electric field in the vacuum with some distribution of charges f.

3.1 Fundamental Solution

The Laplace equation is invariant with respect to all rotations and translations of the euclidean space $\mathbb{R}^n$. Therefore we first look for solutions which are invariant with respect to all rotations. These soutions depend only on the length $r = |x| = \sqrt{x \cdot x}$ of the position vector x. For such functions $u(x) = v(r) = v(\sqrt{x \cdot x})$ we calculate:

$$\nabla_x u(x) = v'\left(\sqrt{x \cdot x}\right) \nabla_x r = v'\left(\sqrt{x \cdot x}\right) \frac{2x}{2r}.$$

Hence the Laplace equation simplifies to an ODE

$$\triangle_x u(x) = \nabla_x \cdot \nabla_x u = v''(r)\frac{x^2}{r^2} + v'(r)\frac{n}{r} - v'(r)\frac{x^2}{r^2 r} = v''(r) + \frac{n-1}{r}v'(r) = 0.$$

Let us solve this ODE:

$$\frac{v''(r)}{v'(r)} = \frac{1-n}{r} \;\Rightarrow\; \ln(v'(r)) = (1-n)\ln(r) + C \;\Rightarrow\; v(r) = \begin{cases} C' \ln(r) + C'' & \text{for } n = 2 \\ \frac{C'}{r^{n-2}} + C'' & \text{for } n \geq 3. \end{cases}$$

Definition 3.1. *Let $\Phi(x)$ be the following solutions of the Laplace equation:*

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\ln|x| & \text{for } n = 2 \\ \frac{1}{n(n-2)\omega_n|x|^{n-2}} & \text{for } n \geq 3. \end{cases}$$

Here ω_n denotes the volume of the unit ball $B(0,1)$ in euclidean space $\mathbb{R}^n$.

This solution has a singularity at the origin $x = 0$. We have chosen the constants in such a way that the following theorem holds:

Theorem 3.2. *For $f \in C_0^2(\mathbb{R}^n)$ a solutions of Poisson's equations $-\triangle u = f$ is given by*

$$u(x) = \int\limits_{\mathbb{R}^n} \Phi(x-y)f(y)d^ny = \int\limits_{\mathbb{R}^n} \Phi(z)f(x-z)d^nz.$$

Proof. The equality of both integrals in the definition of $u(x)$ follows from the substituion $z = x - y$. The second integral is twice continuously differentiable, since f is twice continuously differentiable and has compact support. We calculate

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \int\limits_{\mathbb{R}^n} \Phi(y)\frac{\partial^2 f}{\partial x_i \partial x_j}(x-y)d^ny.$$

In particular, $\triangle u(x) = \int_{\mathbb{R}^n} \Phi(y)\triangle_x f(x-y)d^ny$. We decompose this integral in the sum of an intgeral nearby the singuarity of Φ and an integral away from this singularity:

$$\begin{aligned} \triangle u(x) &= \int\limits_{B(0,\epsilon)} \Phi(y)\triangle_x f(x-y)d^ny && + \int\limits_{\mathbb{R}^n\setminus B(0,\epsilon)} \Phi(y)\triangle_x f(x-y)d^ny \\ &= I_\epsilon && + J_\epsilon. \end{aligned}$$

We use $\int r\ln r dr = \frac{r^2}{2}(\ln r - \frac{1}{2})$ and $\int r dr = \frac{r^2}{2}$ and estimate the first integral for $\epsilon \downarrow 0$:

$$|I_\epsilon| \leq C\|\triangle_x f\|_{L^\infty(\mathbb{R}^n)} \int\limits_{B(0,\epsilon)} |\Phi(y)|d^ny \leq \begin{cases} C\epsilon^2(|\ln\epsilon| + 1) & (n = 2) \\ C\epsilon^2 & (n \geq 3). \end{cases}$$

Integration by parts of the second integral yields

$$\begin{aligned} J_\epsilon &= \int\limits_{\mathbb{R}^n\setminus B(0,\epsilon)} \Phi(y)\triangle_y f(x-y)d^ny \\ &= -\int\limits_{\mathbb{R}^n\setminus B(0,\epsilon)} \nabla_y\Phi(y)\cdot\nabla_y f(x-y)d^ny && + \int\limits_{\partial B(0,\epsilon)} \Phi(y)\nabla_y f(x-y)\cdot N d\sigma(y) \\ &= K_\epsilon && + L_\epsilon. \end{aligned}$$

Here N is the outer normal and $d\sigma$ the measure on the boundary of $\mathbb{R}^n \setminus B(0,\epsilon)$. The second term converges in the limit $\epsilon \downarrow 0$ to zero:

$$|L_\epsilon| \le |\nabla f|_{L^\infty(\mathbb{R}^n)} \int\limits_{\partial B(0,\epsilon)} |\Phi(y)| d\sigma(y) \le \begin{cases} C\epsilon |\ln \epsilon| & (n=2) \\ C\epsilon & (n \ge 3). \end{cases}$$

Another integration by parts of the first term yields

$$\begin{aligned} K_\epsilon &= \int\limits_{\mathbb{R}^n \setminus B(0,\epsilon)} \triangle_y \Phi(y) f(x-y) d^n y - \int\limits_{\partial B(0,\epsilon)} \nabla_y \Phi(y) f(x-y) \cdot N d\sigma(y) \\ &= -\int\limits_{\partial B(0,\epsilon)} \nabla_y \Phi(y) f(x-y) \cdot N d\sigma(y). \end{aligned}$$

Here we used that ϕ is harmonic for $y \neq 0$. The gradient of Φ is equal to $\nabla\Phi(y) = -\frac{1}{n\omega_n}\frac{y}{|y|^n}$. The outer normal points towars the origin and is equal to $-\frac{y}{|y|}$. Let σ_n denote the area of the unit sphere in $\mathbb{R}^n$. Then the volume of the unit ball is

$$\omega_n = \int\limits_0^1 \sigma_n r^{n-1} dr = \frac{\sigma_n}{n}.$$

Now K_ϵ is the mean value of $-f$ on $\partial B(0,\epsilon)$, since $\sigma_n \epsilon^{n-1} = n\omega_n \epsilon^{n-1}$ is the area of $\partial B(0,\epsilon)$. By continuity of f this mean value converges for $\epsilon \downarrow 0$ to $-f(x)$. **q.e.d.**

By this theorem we have $-\triangle\Phi(x) = \delta(x)$ in the sense of distributions. This relation justifies the choice of the constant in the definition of ϕ. The convolution of f with Φ is defined for continuous functions $f \in L^1(\mathbb{R}^n)$. On can show that as a measurable function the convolution is defined even for $f \in L^1(\mathbb{R}^n)$ and belongs to $L^1(\mathbb{R}^n)$. In this case the convolution is in the sense of distributions a solution of Poisson's equation. In general, for a continuous $f \in L^1(\mathbb{R}^n)$ the correpsnding u is not twice differentiable, and the theorem is not valid for such f. But it is valid for all lipschitz continuous functions f in $L^1(\mathbb{R}^n)$. Since Poisson's equation is an inhomogenous lienar PDE, all solutions are defined up to adding a solution of the homogenous equation which is Leplace equation.

3.2 Mean Value Property

In this section we shall prove the following property of a harmonic function u on an open domain $\Omega \subset \mathbb{R}^n$: the value $u(x)$ of u at the center of any ball $B(x,r)$ with compact closure in Ω is equal to the mean of u on the boundary of the ball. Conversely, if this holds for all balls with compact closure in Ω, then u is harmonic. The mean of u on the ball $B(x,r)$ is the mean of the means of u an the boudary of $B(x,r')$ over $r' \in [0,r]$. Therefore the same statement holds for the means of u on the balls $B(x,r)$. This relation is called mean value property and has many important consequences.

Mean Vaule Property 3.3. *Let $u \in C^2(\Omega)$ be harmonic on an open domain $\Omega \subset \mathbb{R}^n$ containing the closure of the ball $B(x,r)$. The mean of u on the ball $B(x,r)$ and on its boundary is equal to the value $u(x)$ of u at the center. Conversely, if the means of a function $u \in C^2(\Omega)$ on all balls $B(x,r)$ with compact closure in Ω or on all boundaries of such balls is equal to the value $u(x)$ at the center of the ball, then u is harmonic.*

Proof. We define $\Phi(r)$ for $x \in \Omega$ as the mean of u on $\partial B(x,r) \subseteq \Omega$:

$$\Phi(r) := \frac{1}{r^{n-1} n\omega_n} \int_{\partial B(x,r)} u(y)\,\mathrm{d}\sigma(y) = \frac{1}{n\omega_n} \int_{\partial B(0,1)} u(x+rz)\,\mathrm{d}\sigma(z).$$

Here ω_n denotes the vloume of the unit ball in euclidean space $\mathbb{R}^n$. We apply the Diveregence Theorem and calculate the derivative $\Phi'(r) =$

$$= \frac{1}{n\omega_n} \int_{\partial B(0,1)} \nabla u(x+rz) \cdot z\,\mathrm{d}\sigma(z) = \frac{1}{r^{n-1} n\omega_n} \int_{\partial B(x,r)} \nabla u(y) \cdot N\,\mathrm{d}\sigma(y) = \frac{1}{r^{n-1} n\omega_n} \int_{B(x,r)} \triangle u\,\mathrm{d}\mu.$$

Hence for harmonic u this function is constant as long as $B(x,r)$ has compact closure in Ω. By continuity of u this function $\Phi(r)$ converges in the limit $\lim r \to 0$ to $u(x)$. This shows that the means of u on all spheres $\partial B(x,r)$ are equal to the vlaues $u(x)$ of u at the center x. Furthermore, the mean of u on the ball $B(x,r)$ is equal to

$$\frac{1}{r^n\omega_n} \int_{B(x,r)} u(y)\,\mathrm{d}^{\mathrm{n}}y = \frac{n}{r^n} \int_0^r \frac{1}{s^{n-1} n\omega_n} \int_{\partial B(x,s)} s^{n-1} u(y)\,\mathrm{d}\sigma(y)\,\mathrm{d}s = \frac{n}{r^n} \int_0^r s^{n-1}\Phi(s)\,\mathrm{d}s.$$

For constant Φ this is again equal to the value $u(x)$ of u at the center x.

Conversely, if the means of u on all balls $B(x,r)$ with compact closure in Ω is equal to the values $u(x)$ of u at the center x, then we have

$$u(x) = \frac{1}{\omega_n r^n} \int_0^r n\omega_n s^{n-1}\Phi(s)\,\mathrm{d}s = \frac{n}{r^n} \int_0^r s^{n-1}\Phi(s)\,\mathrm{d}s.$$

The vanishing of the derivative of the right hand side with respect to r yields

$$-\frac{n^2}{r^{n+1}} \int_0^r s^{n-1}\Phi(s)\,\mathrm{d}s + \frac{n}{r^n} r^{n-1}\Phi(r) = -\frac{n}{r} u(x) + \frac{n}{r}\Phi(r) = 0.$$

Therefore also the means $\Phi(r)$ of u on the boundaries $\partial B(x,r)$ are equal to the value $u(x)$ of u at the center x. Since u is twice continuously differentiable, the function Φ is twice continuously differentiable. We have seen that $\Phi'(r)$ is the mean of $\triangle u$ on $B(x,r)$. In particular, if the mean of u on all $\partial B(x,r) \subset \Omega$ is equal to the value $u(x)$ of u at the center x, then the integral of $\triangle u$ over the balls with compact closure in Ω vanish. If there exists $x \in \Omega$ with $\triangle u(x) \neq 0$, then there is a ball $B(x,r)$ with compact closure in Ω, such that $|\triangle u|$ is larger then $|\triangle u(x)|/2$ on $B(x,r)$ and $\int_{B(x,r)} \triangle u(y) d^n y \neq 0$. Therefore $\triangle u$ vanishes and u is harmonic on Ω. **q.e.d.**

Corollary 3.4. *Let u be a smooth*[1] *harmonic function on an open domain $\Omega \subset \mathbb{R}^n$ and $B(x,r)$ a ball with compact closure in Ω. For all multi-indices α we have the estimate*

$$|\partial^\alpha u(x)| \leq C(n,|\alpha|) r^{-|\alpha|} \|u\|_{L^\infty(\overline{B(x,r)})} \quad \textit{with} \quad C(n,|\alpha|) = 2^{\frac{|\alpha|(1+|\alpha|)}{2}} n^{|\alpha|}.$$

Proof. All partial derivatives of a harmonic function are harmonic. The Mean Value Property and the Divergence Theorem yield for $i = 1, \ldots, n$

$$|\partial_i \partial^\alpha u(x)| = \left| \frac{2^n}{\omega_n r^n} \int_{B(x,r/2)} \partial_i \partial^\alpha u d\mu \right| = \left| \frac{2^n}{\omega_n r^n} \int_{\partial B(x,r/2)} \partial^\alpha u N_i d\sigma \right| \leq \frac{2n}{r} \|\partial^\alpha u\|_{L^\infty(\partial B(x,r/2))}.$$

The inductive application gives first $C(n,1) = 2n$, and using the induction hypothesis

$$\|\partial^\alpha u(y)\| \leq 2^{|\alpha|} C(n,|\alpha|) r^{-|\alpha|} \|u\|_{L^\infty(B(x,r))} \quad \text{for all} \quad y \in \partial B(x,r/2)$$

the relation $C(n,1+|\alpha|) = 2^{1+|\alpha|} n C(n,|\alpha|)$. The given $C(n,|\alpha|)$ is the solution. **q.e.d.**

Liouville's Theorem 3.5. *On $\mathbb{R}^n$ a bounded harmonic function is constant.*

Proof. The foregoing corollary shows that $|\partial_i u(x)|$ is bounded by $2n\|u\|_{L^\infty(\mathbb{R}^n)} r^{-1}$ for each $i = 1, \ldots, n$ and $x \in \mathbb{R}^n$. In the limit $r \to \infty$ the first partial derivatives vanish identically. Therefore u is constant. **q.e.d.**

Let us now transfer the Mean Value Property to a property of distributions. If $u \in C^2(\Omega)$ is harmonic, then for $B(x,r) \subset \Omega$ and $\psi \in C_0^\infty((0,r))$ we have

$$\int_{B(x,r)} u(y) \frac{\psi(|y-x|)}{n|y-x|^{n-1}\omega_n} \mathrm{d}^{\mathrm{n}} y = \int_0^r \frac{\psi(s)}{n s^{n-1} \omega_n} \int_{\partial B(x,s)} u(y) \, \mathrm{d}\sigma(y) \, \mathrm{d}s = \left(\int_0^r \psi(s) \, \mathrm{d}s \right) u(x).$$

So the distribution F_u has the following property:

[1] We shall see in Weyl's Lemma that this assumption can be replaced by the assumption $u \in L^\infty_{\mathrm{loc}}(\Omega)$.

Weak Mean Value Property 3.6. *Let $U \in \mathcal{D}'(U)$ be a harmonic distribution on an open domain $\Omega \subset \mathbb{R}^n$. For each ball $B(x,r)$ with $B(x,r) \subset \Omega$ and each $\psi \in C_0^\infty((0,r))$ with $\int \psi \,\mathrm{d}\mu = 0$ the distribution U vanishes on the following test function:*

$$f \in C_0^\infty(\Omega), \quad y \mapsto f(y) = \frac{\psi(|y-x|)}{n|y-x|^{n-1}\omega_n} \quad \textit{with} \quad \operatorname{supp} f \subset B(x,r) \subset \Omega.$$

Proof. It suffices to show that there exists a test function $g \in C_0^\infty(\Omega)$ with $\triangle g = f$. By the assumptions on ψ there exists a test function $\Psi \in C_0^\infty((0,r))$ with $\Psi' = \psi$. We define

$$g(y) = v(|y-x|) \quad \text{with} \quad v(t) = \int_r^t \frac{\Psi(s)}{ns^{n-1}\omega_n}\,\mathrm{d}s.$$

This function g has compact support in $B(x,r) \subset \Omega$, depends only on $|y-x|$ and is constant on $B(x,\epsilon)$ for some $\epsilon > 0$. We calculate for $y \neq x$:

$$\nabla_y g(y) = v'(|y-x|)\frac{y-x}{|y-x|} \qquad \triangle_y g(y) = v''(|y-x|) + \frac{n-1}{|y-x|}v'(|y-x|)$$

This implies

$$\triangle_y g(y) = \frac{\psi(|y-x|)}{n|y-x|^{n-1}\omega_n} - \frac{(n-1)\Psi(|y-x|)}{n|y-x|^n\omega_n} + \frac{n-1}{|y-x|}\frac{\Psi(|y-x|)}{n|y-x|^{n-1}\omega_n} = f(y). \quad \textbf{q.e.d.}$$

Weyl's Lemma 3.7. *On an open domain $\Omega \subset \mathbb{R}^n$ for each harmonic distribution $U \in \mathcal{D}'(\Omega)$ there exists a harmonic function $u \in C^\infty(\Omega)$ with $U = F_u$.*

Proof. Let us first define u. For all $x \in \Omega$ choose a ball $B(x,r) \subset \Omega$ and a test function $\psi \in C_0^\infty((0,r))$ with $\int_0^r \psi(s)\,\mathrm{d}s = 1$. Then we define

$$u(x) := U(g_x) \quad \text{with} \qquad g_x(y) := \frac{\psi(|y-x|)}{n|y-x|^{n-1}\omega_n}.$$

If U is harmonic, then the Weak Mean Value Property implies that $u(x)$ does not depend on the choice of r and ψ. Hence we can use in the formula for $u(x)$ the same r und ψ for all x in a small neighbourhood of each x_0. Then u is the convolution of the test function $g_0 = \mathsf{P}g_0$ with the distribution U. Due to Lemma 1.9, u is smooth.

Next we prove that the distribution $\tilde{U} = F_u$ has the Weak Mean Value Property. The functions f in the Weak Mean Value Property are characterised by three properties: they depend only on the distance $|y-x|$ of the variable y to some center $x \in \Omega$, they vanish on a neighbourhood of the center x and their integrals vanish. The first property is equivalent to the invariance of f with respect to all rotations around the center. We show that the convolution $g * f$ of a test function g which is invariant with respect to all rotations around the center x with a test function f which is invariant with respect to all rotations around a center z is invariant with respect to all rotations around the center $x + z$. In fact, for all $\mathsf{O} \in SO(n, \mathbb{R})$ we use the invariance of the Lebesgues measure with respect to translations and O and obtain

$$(g*f)(x+z+\mathsf{O}y) = \int_{\mathbb{R}^n} g(x+z+\mathsf{O}y-y')f(y')\,\mathrm{d}^{\mathrm{n}}y' = \int_{\mathbb{R}^n} g(x+\mathsf{O}(y-z'))f(z+\mathsf{O}z')\,\mathrm{d}^{\mathrm{n}}z'.$$

Furthermore, the integral of $g * f$ is the product of the intgerals of g and f:

$$\begin{aligned}\int_{\mathbb{R}^n} (g * f)(x)\,\mathrm{d}^{\mathrm{n}}x = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x-y)f(y)\,\mathrm{d}^{\mathrm{n}}y\,\mathrm{d}^{\mathrm{n}}x &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x-y)f(y)\,\mathrm{d}^{\mathrm{n}}x\,\mathrm{d}^{\mathrm{n}}y \\ &= \left(\int_{\mathbb{R}^n} g(x)\,\mathrm{d}^{\mathrm{n}}x\right)\left(\int_{\mathbb{R}^n} f(y)\,\mathrm{d}^{\mathrm{n}}y\right).\end{aligned}$$

In particular, the convolution $g_0 * f$ of the test functions g_0 with the test function f in the Weak Mean Value Property is again a test function which is invariant with respect to the rotations around the same center as f with vanishing total integral. Since ψ has compact support in $(0, r)$ the function f vanishes on $B(x, \epsilon)$ for sufficiently small $\epsilon > 0$. If the support of g_0 is contained in $B(0, \epsilon/2)$, then $g_0 * f$ vanishes on $B(x, \epsilon/2)$. This implies that $g_0 * f$ is again a function of the form considered in the Weak Mean Value Property and $\tilde{U}(f) = U(g_0 * f) = 0$. So $\tilde{U}$ has the Weak Mean Value Property.

Next we prove that u is harmonic. Let $\phi_{B(0,\epsilon)} \in C_0^\infty(\mathbb{R})$ be the mollifier defined in Section 1.4. For any $B(x, r)$ with compact closure in Ω, there exists $R > r$ with $B(x, R) \subset \Omega$. For all $0 < r_1 < r_2 < R$ and sufficiently small ϵ the function $\psi(t) = \phi_{B(0,\epsilon)}(t-r_1)-\phi_{B(0,\epsilon)}(t-r_2)$ has compact support in $(0, R)$ and vanishing total integral. Let f_ϵ denote the corresponding functions in the kernel of $\tilde{U}$. Since u is continuous, the limit $\epsilon \downarrow 0$ of $\tilde{U}(f_\epsilon)$ exists and converges to the difference of the means of u on $\partial B(x, r_2)$ and $\partial B(x, r_1)$. Since $\tilde{U}$ has the Weak Mean Value Property these differences vanish for all $0 < r_1 < r_2 < R$ and the means of u on all $\partial B(x, r) \subset \Omega$ coincide. In the limit $r \downarrow 0$ these means converge to $u(x)$, since u is continuous. Therefore u has the Mean Value Property and is a smooth harmonic function.

Finally we prove $\tilde{U} = U$. The functions $\psi(t) = \phi_{B(0,\epsilon/3)}(t - 2/3\epsilon)$ have support $[\epsilon/3, \epsilon]$ and total intgeral 1. The corresponding functions g_0 are smooth mollifiers λ_ϵ. By definition of $\tilde{U}$ we have $\tilde{U} = \lambda_\epsilon * U$. Now Lemma 1.10 implies $\tilde{U} = U$. **q.e.d.**

Actually we have proven that any distribution U which has the Weak Mean Value Property corresponds to a smooth harmonic function. Therefore the weak solutions of the Laplace equations coincide with the strong solutions, and all solutions are smooth.

Let us finish this section with a proof of Harnack's inequality. This inequality estimates the values of a positive harmonic on a path–connected domain in terms of the value at any other point in the domain.

Harnack's Inequality 3.8. *Let Ω' be an open path–connected domain with compact closure in the open domain $\Omega \subset \mathbb{R}^n$. Then there exists a costant $C > 0$ depending only on Ω' and Ω, such that any non-negative harmonic function u on Ω satisfies the Harnack inequality*

$$\sup_{x\in\Omega'} u(x) \leq C \inf_{x\in\Omega'} u(x).$$

In particular, for all $x, y \in \Omega'$ we have $$\frac{1}{C}u(y) \leq u(x) \leq Cu(y).$$

Proof. Let r be the maximal value of the continuous function

$$r : \Omega' \to \mathbb{R}^+, \quad x \mapsto r(x) = \sup\{r > 0 \mid B(x, 2r) \subset \Omega\}.$$

For $x \in \Omega'$ and $y \in B(x, r)$ we have $B(y, r) \subset B(x, 2r) \subset \Omega$. The Mean Value Property implies

$$u(x) = \frac{1}{2^n r^n \omega_n} \int_{B(x,2r)} u \,\mathrm{d}\mu \geq \frac{2^{-n}}{r^n \omega_n} \int_{B(y,r)} u \,\mathrm{d}\mu = 2^{-n} u(y).$$

Since $\bar{\Omega}'$ is compact and path–connceted, it can be covered by finitley many balls $B_1, \ldots, B_N$ of radius $\frac{r}{2}$ such that $B_{n+1} \cap B_n \neq \emptyset$ for $n = 1, \ldots, N-1$. An N-fold application of the special case implies for general $x, y \in \Omega'$

$$u(x) \geq 2^{-nN} u(y).$$

Taking the infimum over $x \in \Omega'$ and the supremum over $y \in \Omega'$ gives

$$\sup_{x\in\Omega'} u(x) \leq 2^{nN} \inf_{x\in\Omega'} u(x).$$ **q.e.d.**

Harnack's Principle 3.9. *On an open and path–connected domain $\Omega \subset \mathbb{R}^n$ a monotone sequence of hamonic functions $(u_n)_{n\in\mathbb{N}}$ converges uniformly on all compact subsets if and only if there exists $x \in \Omega$ such that $(|u_n(x)|)_{n\in\mathbb{N}}$ is bounded.*

Proof. If $(u_n)_{n\in\mathbb{N}}$ converges uniformly on compact subsets, then $(u_n(x))_{n\in\mathbb{N}}$ converges for all $x \in \Omega$. Conversely, let $(|u_n(x)|)_{n\in\mathbb{N}}$ be bounded for some $x \in \Omega$. By monotonicity the sequence $(u_n(x))_{n\in\mathbb{N}}$ converges. Furthermore, we may assume that $(u_n - u_m)_{n\geq m}$ is non-negative. Harnack's Inequality implies that $(u_n - u_m)_{n\geq m}$ is uniformly bounded on compact subsets of Ω and monotonic. Hence it converges uniformly there. The limit has together with all u_n the Mean Value Property and is harmonic. **q.e.d.**

3.3 Maximum Principle

Let the harmonic function u take in a point x of an open path–connected domain $\Omega \subset \mathbb{R}^n$ a maximum. The Mean Value Property implies on all balls $B(x,r) \subset \Omega$

$$\frac{1}{r^n\omega_n}\int\limits_{B(x,r)} |u(x)-u(y)|d^ny = \frac{1}{r^n\omega_n}\int\limits_{B(x,r)} (u(x)-u(y))d^ny = 0.$$

Hence u takes the maximum on all these balls $B(x,r) \subset \Omega$. Since Ω is path–connected every other point $y \in \Omega$ is conneceted with x by a continuous path $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$. The compact image $\gamma[0,1]$ is covered by finitly many balls $B(\gamma(t_1), r_1), \ldots, B(\gamma(t_N), r_N) \subset \Omega$ with $0 \le t_1 < \ldots t_N \le 1$ and $r_1, \ldots, r_N > 0$. Hence u is constant along γ, and on Ω since this is true for all $y \in \Omega$. This proves

Strong Maximum Principle 3.10. *If a harmonic function u has on a path–connected open domain $\Omega \subset \mathbb{R}^n$ a maximum, then u is constant.* **q.e.d.**

Weak Maximum Principle 3.11. *Let the harmonic function u on a bounded open domain $\Omega \subset \mathbb{R}^n$ extend continuously to the boundary $\partial\Omega$. The maximum of u is taken on the boundary $\partial\Omega$.*

Proof. By Heine Borel the closure $\bar{\Omega}$ is compact and the continuous function u takes on $\bar{\Omega}$ a maximum. If it does not belong to $\partial\Omega$, then u is constant on the corresponding path–connected component and the maximum is also taken on $\partial\Omega$. **q.e.d.**

Since the negative of a harmonic function is harmonic the same conclusion holds for minima. Now we generalise the Maximum Principle, but not the Mean Value Property.

Definition 3.12. *On an open domain $\Omega \subset \mathbb{R}^n$ an differential operator L*

$$Lu = \sum\nolimits_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum\nolimits_{i=1}^n b_i(x)\frac{\partial u(x)}{\partial x_i}$$

with symmetric coefficients $a_{ij} = a_{ji}$ is called elliptic, if

$$\sum\nolimits_{i,j=1}^n a_{ij}(x)k_ik_j > 0 \quad \textit{for alle } x \in \Omega \textit{ and all } k \in \mathbb{R}^n \setminus \{0\}.$$

If we replace a_{ij} by $\frac{1}{2}(a_{ij} + a_{ji})$, then the assumption $a_{ij} = a_{ji}$ is always fullfilled. Due to the commutativity of the secod derivatives this replacement does not change L.

Theorem 3.13. *Let L be an elliptic operator on a bounded open domain $\Omega \subset \mathbb{R}^n$ whose coefficients a_{ij} und b_i extend continuously and elliptic to $\partial\Omega$. Every twice differentiable solution u of $Lu \ge 0$ which extends continuously to $\partial\Omega$ takes the maximum on $\partial\Omega$.*

Proof. Let us first show that L is uniform eliptic, i.e. there exists $\lambda > 0$ with

$$\sum_{i,j=1}^{n} a_{ij}(x)k_ik_j \geq \lambda \sum_{i=1}^{n} k_i^2 \quad \text{for all } x \in \Omega \text{ and all } k \in \mathbb{R}^n.$$

The continuous function $(x,k) \mapsto \sum_{i,j=1}^{n} a_{ij}(x)k_ik_j$ attains on the compact set $(x,k) \in \bar{\Omega} \times S^{n-1} \subset \bar{\Omega} \times \mathbb{R}^n$ a minimum $\lambda > 0$. Hence L is uniform elliptic.

For $v(x) = \exp(\alpha x_1)$ mit $\alpha > 0$ we conclude

$$Lv = \alpha(\alpha a_{11}(x) + b_1(x))v \geq \alpha(\alpha\lambda + b_1(x))v.$$

The continuous coefficients b_i are bounded on the compact set $\bar{\Omega}$. Therfore there exists $\alpha > 0$ with $Lv > 0$. By linearity of L we obatin $L(u + \epsilon v) > 0$ on Ω for all $\epsilon > 0$. The continuous functions $u+\epsilon v$ attains on $\bar{\Omega}$ a maximum. The first derivative of the function $u+\epsilon v$ which is twice differentiable on Ω vanishes at a maximum $x_0 \in \Omega$ and the Hessian is negative semi-definite. In particular there exisist an orthogonal matrix $B \in O(n)$ and non-positive $\lambda_1, \ldots, \lambda_n$ with $\frac{\partial^2(u+\epsilon v)(x_0)}{\partial x_i \partial x_j} = \sum_k B_{ki}\lambda_k B_{kj}$. Now the ellipticity implies $-L(u + \epsilon v)(x_0) \geq -\lambda \sum_{ki} \lambda_k B_{ki}^2 \geq 0$ and contradicts to $L(u + \epsilon v) > 0$. Therefore for all $\epsilon > 0$ the maximum belongs to the boundary $x_0 \in \partial\Omega$:

$$\sup_{x\in\Omega} u(x) + \epsilon \inf_{x\in\Omega} v(x) \leq \sup_{x\in\Omega}(u(x) + \epsilon v(x)) = \max_{x\in\partial\Omega}(u(x) + \epsilon v(x)) \leq \max_{x\in\partial\Omega} u(x) + \epsilon \max_{x\in\partial\Omega} v(x).$$

Because this holds for all $\epsilon > 0$ the boundedness of v on $\bar{\Omega}$ implies the theorem. **q.e.d.**

The negative of the functions u in the theorem obey $Lu \leq 0$ and take a minimum on the boundary. In particular, the solutions u of $Lu = 0$ take the maximum and the minimum on the boundary.

3.4 Green's Function

In this section we try to find some conditions which ensure the existence and uniqueness of a harmonic function on an path–connected, open and bounded domain $\Omega \subset \mathbb{R}^n$. A natural candidate for further conditions are boundary value problems. This means that we assume that either the harmonic function or some of its derivatives extends continuously to the boundary and coincides there with a given function. We call a function u on the closure $\bar{\Omega}$ of an domain m times continuously differentiable, if it is m times continuously differetialble on Ω and all partial derivatives of order at most m extend continuously to $\partial\Omega$.

In the following formula we apply The Diveregnce Theorem to $x \mapsto v(x)\nabla u(x)$:

First Green's Formula 3.14. *Let the Divergence Theorem hold on the open and bounded domain $\Omega \subset \mathbb{R}^n$. Then for two functions $u, v \in C^2(\bar{\Omega})$ we have*

$$\int_{\Omega} v(y)\triangle u(y) d^n y + \int_{\Omega} \nabla v(y) \cdot \nabla u(y) d^n y = \int_{\partial\Omega} v(z) \nabla u(z) \cdot N d\sigma(z). \qquad \textbf{q.e.d.}$$

If we substract the formula for interchanged u and v, then we obtain:

Second Green's Formula 3.15. *Let the Divergence Theorem hold on the open and bounded domain $\Omega \subset \mathbb{R}^n$. Then for two functions $u, v \in C^2(\bar{\Omega})$ we have*

$$\int_{\Omega} \left(v(y)\triangle u(y) - u(y)\triangle v(y) \right) d^n y = \int_{\partial\Omega} \left(v(z)\nabla u(z) - u(z)\nabla v(z) \right) \cdot N d\sigma(z). \quad \textbf{q.e.d.}$$

Let us apply this formula to the fundamental solution $v(y) = \Phi(x-y)$. This solution is harmonic only for $y \neq x$. Like in the proof of Theorem 3.2 we restrict the integral to the complment of $B(x, \epsilon)$. For $u \in C^2(\mathbb{R}^n)$ we showed in the proof of Theorem 3.2 the following estimate:

$$\lim_{\epsilon \to 0} \int_{\partial(\mathbb{R}^n \setminus B(x,\epsilon))} u(z) \nabla_z \Phi(x-z) \cdot N d\sigma(z) = \lim_{\epsilon \to 0} \int_{\partial(\mathbb{R}^n \setminus B(0,\epsilon))} u(x-z) \nabla_z \Phi(z) \cdot N d\sigma(z) = u(x).$$

In the limit $\epsilon \to 0$ all other integrals converge to zero. This proves

Green's Representation Theorem 3.16. *Let the Divergence Theorem hold on the open and bounded domain $\Omega \subset \mathbb{R}^n$. Then for $x \in \Omega$ and a function $u \in C^2(\bar{\Omega})$ we have*

$$u(x) = -\int_{\Omega} \Phi(x-y) \triangle u(y) d^n y + \int_{\partial\Omega} \left(\Phi(x-z) \nabla_z u(z) - u(z) \nabla_z \Phi(x-z) \right) \cdot N d\sigma(z).$$

This implies that on Ω each solution of the Poisson equation is uniquely determined by the vlaues of u and the normal derivative $\nabla u \cdot N$ on $\partial\Omega$. Conversely, we look for such functions, such that there exists a solution of Poisson's equation with the addtional condition that u and $\nabla u \cdot N$ take on $\partial\Omega$ the given values. The Weak Maximum Principle implies the the harmonic function is already uniquely determined by the values of u on $\partial\Omega$. So we formulate the following boundary value problem:

Dirichlet Problem 3.17. *For a given function f on an open domain $\Omega \subset \mathbb{R}^n$ and g on $\partial\Omega$ we look for a solution u of $-\triangle u = f$ on Ω which extends continuously to $\partial\Omega$ and coincides there with g.*

Green's Function 3.18. *A function $G_\Omega : \{(x,y) \in \Omega \times \Omega \mid x \neq y\} \to \mathbb{R}$ is called Green's function for the open domain $\Omega \subset \mathbb{R}^n$, if it has the following two properties:*

(i) *For $x \in \Omega$ the function $y \mapsto G_\Omega(x,y) - \Phi(x-y)$ is harmonic on $y \in \Omega$.*

(ii) *For $x \in \Omega$ the function $y \mapsto G_\Omega(x,y)$ extends continuously to $\partial\Omega$ and vanishes on $y \in \partial\Omega$.*

Green's Second Formula yields for the function $v(y) = G_\Omega(x,y) - \Phi(x-y)$:

$$-\int\limits_{\Omega} \Phi(x-y)\triangle u(y)d^n y + \int\limits_{\partial\Omega} (\Phi(x-z)\nabla_z u(z) - u(z)\nabla_z \Phi(x-z)) \cdot N d\sigma(z)$$
$$= -\int\limits_{\Omega} G_\Omega(x,y)\triangle u(y)d^n y - \int\limits_{\partial\Omega} u(z)\nabla_z G_\Omega(x,z) \cdot N d\sigma(z).$$

Now Geen's Representation Theorem implies

$$u(x) = -\int\limits_{\Omega} G_\Omega(x,y)\triangle_y u(y)d^n y - \int\limits_{\partial\Omega} u(z)\nabla_z G_\Omega(x,z) \cdot N d\sigma(z).$$

If, conversely, the functions $f : \bar{\Omega} \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ have sufficient regularity, then

$$u(x) = \int\limits_{\Omega} G_\Omega(x,y)f(y)d^n y - \int\limits_{\partial\Omega} g(z)\nabla_z G_\Omega(x,z) \cdot N d\sigma(z)$$

solves the Dirichlet Problem. In fact by Theorem 3.2 the first term solves the Dirichlet Problem for $g = 0$. If $g : \partial\Omega \to \mathbb{R}$ is the boundary value of a function on Ω with sufficient regularity, then the difference of g minus the correpsonding first term is harmonic and coincides with the second term. Therfore the Dirichlet Problem reduces to the search of the Green's Function.

For $x \in \Omega$ the difference $y \mapsto G_\Omega(x,y) - \Phi(x-y)$ is harmonic on $y \in \Omega$ and on the boundary $\partial\Omega$ equal to $-\Phi(x-y)$. Hence the difference is the solution of the Dirichlet Problem for $f = 0$ and $g(x) = -\Phi(x-y)$.

Theorem 3.19 (Symmetry of the Green's Function)**.** *If there is a Green's Function G_Ω for the domain Ω, then $G_\Omega(x,y) = G_\Omega(y,x)$ holds for all $x \neq y \in \Omega$.*

Proof. For $x \neq y \in \Omega$ let $\epsilon > 0$ be sufficiently small, such that both balls $B(x,\epsilon)$ und $B(y,\epsilon)$ are disjoint subsets of Ω. Green's Second Formula implies for the domain

$\Omega \setminus (B(x,\epsilon) \cup B(y,\epsilon))$ and the functions $u(z) = G(x,z)$ and $v(z) = G(y,z)$

$$\int\limits_{\partial B(x,\epsilon)} (G(y,z)\nabla_z G(x,z) - G(x,z)\nabla_z G(y,z)) \cdot N d\sigma(z)$$
$$= \int\limits_{\partial B(y,\epsilon)} (G(x,z)\nabla_z G(y,z) - G(y,z)\nabla_z G(x,z)) \cdot N d\sigma(z).$$

For $\epsilon \to 0$ the estimate for L_ϵ in the proof of Theorem 3.2 shows that both second terms converge to zero. The calculaton of K_ϵ in the proof of Theorem 3.2 carries over and shows that the first terms converge to $G(y,x)$ and $G(x,y)$, respectively. **q.e.d.**

We shall calculate Green's function for all balls in $\mathbb{R}^n$. Let us first restrict to the unit ball $\Omega = B(0,1)$. We may use the inversion $x \mapsto \tilde{x} = \frac{x}{|x|^2}$ at the unit sphere $\partial B(0,1)$ in order to solve the corresponding Dirichlet Problem. The inversion maps the inside of the unit ball to the outside and vice versa. It helps to solve the Dirichlet Problem for $f = 0$ and $g(x) = \Phi(x - y)$:

Green's Function of the unit ball 3.20. *The Green's Function of $B(0,1)$ is*

$$G_{B(0,1)}(x,y) = \Phi(x-y) - \Phi(|x|(\tilde{x}-y)) = \begin{cases} \Phi(x-y) - |x|^{2-n}\Phi(\tilde{x}-y) & \text{for } n > 2 \\ \Phi(x-y) - \Phi(\tilde{x}-y) - \Phi(x) & \text{for } n = 2. \end{cases}$$

Proof. For $|y| = 1$ we have $|x|^2|\tilde{x} - y|^2 = 1 - 2y \cdot x + |x|^2 = |x - y|^2$. Hence $\Phi(|x|(\tilde{x} - y))$ and $\Phi(x - y)$ coincide on the boundary $y \in \partial B(0,1)$. **q.e.d.**

Poisson's Representation Formula 3.21. *For $f \in C^2(\overline{B(z,r)})$ and $g \in C(\partial B(z,r))$ the unique solution of the Dirichlet Problem on $\Omega = B(z,r)$ is given by*

$$u(x) = \frac{1}{r^{n-2}} \int\limits_{B(z,r)} G_{B(0,1)}(\tfrac{x-z}{r}, \tfrac{y-z}{r}) f(y) d^n y + \frac{1 - \frac{|x-z|^2}{r^2}}{n\omega_n} \int\limits_{\partial B(0,1)} \frac{g(z+ry)}{|\frac{x-z}{r} - y|^n} d\sigma(y).$$

Proof. The affine map $x \mapsto \frac{x-z}{r}$ is a homoemorphism from $B(z,r)$ onto $B(0,1)$ and from $\partial B(z,r)$ onto $\partial B(0,1)$. The difference $r^{2-n}\Phi(\frac{x-z}{r} - \frac{y-z}{r}) - \Phi(x-y)$ vanishes for $n > 2$ and is constant for $n = 2$. Therefore the Green's function of $B(z,r)$ is equal to

$$G_{B(z,r)}(x,y) = r^{2-n} G_{B(0,1)}(\tfrac{x-z}{r}, \tfrac{y-z}{r}).$$

It suffices to consider the two cases $g = 0$ and $f = 0$ separately. The properties of the Green's function together with Theorem 3.2 show, that for $g = 0$ the function u

differes by a harmonic function from a solution of Poisson's equation. By the symmetry of the Green's Function the map $x \mapsto G_{B(z,r)}(y,x)$ extends continuously to $\overline{B(z,r)}$ and vanishes on the the boundary $x \in \partial B(z,r)$. This finishes the proof for $g = 0$.

For $|y| = 1$ and $n > 2$ we observe (the reader should check this formula for $n = 2$):

$$\begin{aligned}
K(x,y) &= -\nabla_y G_{B(0,1)}(x,y) \cdot \frac{y}{|y|} = -\nabla_y G_{B(0,1)}(y,x)\frac{y}{|y|} \\
&= -\frac{1}{n(n-2)\omega_n}\frac{y}{|y|}\nabla_y\left(\frac{1}{|y-x|^{n-2}} - \frac{1}{|y|^{n-2}\,|\tilde{y}-x|^{n-2}}\right) \\
&= \frac{1}{n\omega_n}\frac{y}{|y|}\left(\frac{y-x}{|y-x|^n} - \frac{y}{|y|^n\,|\tilde{y}-x|^{n-2}} - \frac{(\tilde{y}-x)\left(\frac{1}{|y|^2} - 2\frac{y^2}{|y|^4}\right)}{|y|^{n-2}\,|\tilde{y}-x|^n}\right) \\
&= \frac{1}{n\omega_n}\frac{1-xy-(|x|^2-2xy+1)+(1-xy)}{|x-y|^n} = \frac{1-|x|^2}{n\omega_n|x-y|^n}.
\end{aligned}$$

By the Symmetry of the Green's Function the function $x \mapsto K(x,y)$ is harmonic. Hence for $f = 0$ the given function u is harmonic. For finishing the proof we show that

$$u(z+rx) = \int\limits_{\partial B(0,1)} g(z+ry)K(x,y)d\sigma(y)$$

extends continuously to $x \in \partial B(0,1)$ and coincides there with $g(z+rx)$. We observe

(i) the integral kernel $K(x,y)$ is for $(x,y) \in B(0,1) \times \partial B(0,1)$ positiv.

(ii) For all $x \in \partial B(0,1)$ and $\epsilon > 0$ the family of functions $y \mapsto K(\lambda x, y)$ converge for $\lambda \uparrow 1$ on $y \in \partial B(0,1) \setminus B(x,\epsilon)$ uniformly to zero, and

(iii) Green's Second Formula for the function $v(y) = G_{B(0,1)}(x,y) - \Phi(x-y)$ and $u = 1$ and Green's Representation Formula for $u = 1$ yields

$$\int\limits_{\partial B(0,1)} K(x,y)d\sigma(y) = 1 \quad \text{for} \quad x \in B(0,1).$$

For continuous g the properties (i)-(iii) ensure that in the limit $\lambda \uparrow 1$ the family of functions $x \mapsto \int_{\partial B(0,1)} g(y)K(\lambda x,y)d\sigma(y)$ converge on $\partial B(0,1)$ uniformly to g. **q.e.d.**

A harmonic function u on $B(z,r)$ which extends continuously to $\partial B(z,r)$ obeys

$$u(x) = \frac{1-\frac{|x-z|^2}{r^2}}{n\omega_n}\int\limits_{\partial B(0,1)} \frac{u(z+ry)}{|\frac{x-z}{r}-y|^n}d\sigma(y) = \frac{r^2-|x-z|^2}{nr\omega_n}\int\limits_{\partial B(z,r)} \frac{u(y)}{|x-y|^n}d\sigma(y).$$

In particular, u is completely determined by the vlaues on $\partial B(z,r)$. Partial derivatives with respect to x yield similar formulas for the values of all partial derivatives of u. This formula implies the Mean Value property. For all $y \in \partial B(z,r)$ the Taylor series of $x \mapsto |x-y|^{-n} = (y^2 - 2xy + x^2)^{-\frac{n}{2}}$ in $x = z$ converge on all balls $B(z,r')$ with $r' < r$ uniformly to $|x-y|^{-n}$. Consequently all harmonic functions are analytic.

Corollary 3.22. *Harmonic functions on an open domain $\Omega \subset \mathbb{R}^n$ are analytic.***q.e.d.**

Exercise 3.23. 1. *Show the estimate $|\partial^\alpha |x|^{-n}| \leq C(n,|\alpha|)|x|^{-n-|\alpha|}$ for all $|x| \neq 0$ and all multi-indices α with a constant $C(n,|\alpha|)$ depending only on n and $|\alpha|$.*

2. *Give another proof of Corollary 3.4.*

Lemma 3.24. *Let $\Omega \subset \mathbb{R}^n$ be an open neighbourhood of 0 and u a bounded harmonic function on $\Omega \setminus \{0\}$. Then u extends as a harmonic function to Ω.*

Proof. On a ball $B(0,r)$ with compact closure in Ω Theorem 3.21 gives a harmonic function $\tilde{u}$ which coincdies on $\partial B(0,r)$ with u. The family of harmonic functions $u_\epsilon(x) = \tilde{u}(x) - u(x) + \epsilon G_{B(0,r)}(x,0)$ on $B(0,r) \setminus \{0\}$ vanish on $\partial B(0,r)$. If for any $\epsilon > 0$ the function u_ϵ takes on $B(0,r) \setminus \{0\}$ a negative value, then due to the boundedness of u and $\tilde{u}$ and the unboundedness of $G_{B(0,r)}(\cdot,0)$ the harmonic function u_ϵ has a negative minimum on $B(0,r) \setminus \{0\}$. This contradicts the Strong Maximum Principle. Hence u_ϵ is non-negative. Analogously u_ϵ us for negative ϵ non-positive. Otherwise u_ϵ would have a positive maximum in $B(0,r) \setminus \{0\}$. In the limit $\epsilon \to 0$ $u_0 = \tilde{u} - u$ vanishes identically and $\tilde{u}$ is a harmonic extension of u to Ω. **q.e.d.**

The proof shows a slightly stronger statement. Each harmonic function on $\Omega \setminus \{0\}$ whose absolute value $|u(x)|$ is for all $\epsilon > 0$ bounded by $\epsilon G_{B(0,r)}(x,0)$ on $B(0,\delta) \setminus \{0\}$ with sufficiently small $\delta > 0$ depending on ϵ has an harmonic extension to Ω.

3.5 Dirichlet's Principle

The unique solution of Dirichlet's Problem solves also a variational problem.

Dirichlet's Prinzip 3.25. *Let $\Omega \subset \mathbb{R}^n$ be bounded and open and obey the assumptions of the Divergence Theorem. For continuous real functions f on $\bar{\Omega}$ and g on $\partial\Omega$ the solution u of the Dirichlet Problem 3.4 is the minimizer of the following functional:*

$$I : \{w \in C^2(\bar{\Omega}) \mid w|_{\partial\Omega} = g\} \to \mathbb{R}, \qquad w \mapsto I(w) = \int\limits_\Omega \left(\frac{1}{2}\nabla w \cdot \nabla w - wf\right) d^n x.$$

Proof. Let u be a solution of the Dirichlet Problem and w another function in the domain $\{w \in C^2(\bar{\Omega}) \mid w|_{\partial\Omega} = g\}$ of I. An integration by parts yields

$$0 = \int\limits_{\Omega} (-\triangle u - f)(u - w) d^n x = \int\limits_{\Omega} \left(\nabla u \cdot \nabla(u - w) - f(u - w)\right) d^n x.$$

$$\begin{aligned}\int\limits_{\Omega} (\nabla u \cdot \nabla u - fu)\, d^n x &= \int\limits_{\Omega} (\nabla u \cdot \nabla w - fw)\, d^n x \leq \\ &\leq \int\limits_{\Omega} \frac{1}{2} \nabla u \cdot \nabla u d^n x + \int\limits_{\Omega} \left(\frac{1}{2} \nabla w \cdot \nabla w - fw\right) d^n x\end{aligned}$$

Here we used the Cauchy-Schwarz inequality:

$$\begin{aligned}\int\limits_{\Omega} \nabla u \cdot \nabla w d^n x \leq \int\limits_{\Omega} \nabla u \cdot \nabla w d^n x + \frac{1}{2} \int\limits_{\Omega} (\nabla u - \nabla w) \cdot (\nabla u - \nabla w) d^n x = \\ \int\limits_{\Omega} \frac{1}{2} \nabla u \cdot \nabla u d^n x + \int\limits_{\Omega} \frac{1}{2} \nabla w \cdot \nabla w d^n x.\end{aligned}$$

This shows $I(u) \leq I(w)$.

If, conversely, u is a minimum, then all $v \in C^2(\bar{\Omega})$ which vanish on $\partial\Omega$ obey

$$\begin{aligned}0 = \frac{d}{dt} I(u + tv)\Big|_{t=0} &= \frac{d}{dt} \left(I(u) + t \int\limits_{\Omega} (\nabla u \cdot \nabla v - fv)\, d^n x + \frac{t^2}{2} \int\limits_{\Omega} \nabla v \cdot \nabla v d^n x \right)\Bigg|_{t=0} \\ &= \int\limits_{\Omega} (\nabla u \cdot \nabla v - fv)\, d^n x = \int\limits_{\Omega} (-\triangle u - f) v d^n x.\end{aligned}$$

The final integration by parts shows $-\triangle u = f$ on Ω. **q.e.d.**

Finally we remark that one can also prove the uniqueness of the solution with the help of this functional. The difference of two solutions solves the Dirichelte Problem for $f = 0$ and $g = 0$. In this case the functional is non-negative, and vanishes if and only if u is constant. The boundary conditions forces this constant to be zero. By using this principle one can show in a larger class of functions, that this functional has a unique minimizer, which thereby solves the Dirichlet Problem.