Chapter 2

First order PDE's

2.1 Homogenous Transport Equation

One of the simplest partial differential equations is the transport equation:

$$\dot{u} + b \cdot \nabla u = 0.$$

Here $b \in \mathbb{R}^n$ is a vector and $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is the unknown function. The product $b \cdot \nabla u$ denotes the scalar product of the vector b with the vector of the first partial derivatives of u with respect to x:

$$b \cdot \nabla u(x,t) = b_1 \frac{\partial u(x,t)}{\partial x_1} + \ldots + b_n \frac{\partial u(x,t)}{\partial x_n}.$$

Let us first assume that $u(x,t)$ is a differentiable solution of the transport equation. For all fixed $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ the function

$$z(s) = u(x + s \cdot b, t + s)$$

is a differentiable function on $s \in \mathbb{R}$, whose first derivative vanishes:

$$\dot{z}(s) = b \nabla u(x + s \cdot b, t + s) + \dot{u}(x + s \cdot b, t + s) = 0.$$

Therefore u is constant along all parallel stright lines in direction of $(b,1)$. Furthermore, u is completely determined by the values on all these parallel straigth lines.

Initial Value Problem 2.1. *We are looking for a solution $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ of the transport equation with given b, which at $t = 0$ is equal to some given function $g : \mathbb{R}^n \to \mathbb{R}$.*

All parallel stright lines in direction of $(b,1)$ intersect $\mathbb{R}^n \times \{0\}$ exactely once:

$$(x+sb, t+s) \in \mathbb{R}^n \times \{0\} \iff s = -t$$

Hence the soluton has to be equal to $u(x,t) = g(x-tb)$. If g is differentialble on $\mathbb{R}^n$, then this function indeed solves the transport equation. In this case the initial value problem has a unique solution.

Otherwise, if g is not differentialble on $\mathbb{R}^n$, then the initial value problem does not have a solution. Furthermore, as we have seen above, whenever the initial value problem has a solution, then the function $u(x,t) = g(x-bt)$ is the solution. It can be shown that this is also true for weak solutions. For example if the function g corresponds to a distribution $F_g \in \mathcal{D}'(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} g(x-tb)\phi(x,t)\, \mathrm{d}^n x\, \mathrm{d}t = \int_{\mathbb{R}^n} \int_{\mathbb{R}} g(x-tb)\phi(x,t) d\mu(t)\, \mathrm{d}^n x =$$
$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}} g(y)\phi(y+bt)\, \mathrm{d}t\, \mathrm{d}^n y = F_g\left(\int_{\mathbb{R}} \mathsf{T}_{-tb}\phi(\cdot,t)\, \mathrm{d}t\right).$$

Since $C_0^\infty(\mathbb{R}^n \times \mathbb{R}) \to C_0^\infty(\mathbb{R}^n)$ with $\phi \mapsto \int_{\mathbb{R}} \mathsf{T}_{-tb}\phi(\cdot,t)\, \mathrm{d}t$ is continuous and linear, the function $g(x-bt)$ corresponds to a distribution on $\mathbb{R}^n \times \mathbb{R}$. Let us now consider a distribution $F \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ which solves the transport equation $(\partial_t + b\nabla)F = 0$. The following distribution solves the equation $\partial_t \tilde{F} = 0$:

$$\tilde{F} \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}) \text{ with } \tilde{F}(\phi) = F(\tilde{\phi}) \text{ and } \tilde{\phi}(y,t) = \phi(y+bt,t) \text{ for all } (y,t) \in \mathbb{R}^n \times \mathbb{R}.$$

The following exercise shows that there exists a unique distribution $G \in \mathcal{D}'(\mathbb{R}^n)$ with

$$\tilde{F}(\phi) = G\left(\int_{\mathbb{R}} \phi(\cdot,t)\, \mathrm{d}t\right) \quad \text{and} \quad F(\phi) = G\left(\int_{\mathbb{R}} \mathsf{T}_{-tb}\phi(\cdot,t)\, \mathrm{d}t\right).$$

Exercise 2.2. *1. Show that the following formula defines a linear continuous map*

$$I : C^\infty(\mathbb{R}^n \times \mathbb{R}) \to C_0^\infty(\mathbb{R}^n) \quad \textit{with} \quad I(\phi)(x) = \int\limits_{\mathbb{R}} \phi(x,t)\, \mathrm{d}t.$$

2. Let $\tilde{F} \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ solve the equation $\partial_t \tilde{F} = 0$. Show that there exists a distribution $G \in \mathcal{D}'(\mathbb{R}^n)$, such that $\tilde{F}(\phi) = G(I(\phi))$.

3. Show that for any mollifier $(\lambda_\epsilon)_{\epsilon>0}$ on $\mathbb{R}$ and any $\phi \in C_0^\infty(\mathbb{R}^n)$ the functions

$$\phi \times \lambda_\epsilon : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \quad \textit{with} \quad (x,t) \mapsto \phi(x)\lambda_\epsilon(t)$$

belong to $C_0^\infty(\mathbb{R}^n \times \mathbb{R})$ and that $\tilde{F}(\phi \times \lambda_\epsilon)$ does not depend on $\epsilon > 0$. Furthermore, the map $\phi \mapsto \tilde{F}(\phi \times \lambda_\epsilon)$ defines the distribution $G \in \mathcal{D}'(\mathbb{R}^n)$ above. We may interpret the distribution $G \in \mathcal{D}'(\mathbb{R}^n)$ as the inital value $G(\phi) = \lim_{\epsilon \downarrow 0} \tilde{F}(\phi \times \lambda_\epsilon)$, which uniquely determines $\tilde{F}$ and F.

2.2 Inhomogenous Transport Equation

Now we consider the corresponding inhomogenous trasport equation:

$$\dot{u} + b \cdot \nabla u = f.$$

Again $b \in \mathbb{R}^n$ is a given vector, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a given function and $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is the unknown function.

Initial Value Problem 2.3. *We are looking for a given vector $b \in \mathbb{R}$, a given function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and a given initial value $g : \mathbb{R}^n \to \mathbb{R}$ for a solution $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ of the inhomogenous transport equation which is at $t = 0$ equal to g.*

We define for each $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ the function $z(s) = u(x + sb, t + s)$ which solves

$$\dot{z}(s) = b \cdot \nabla u(x + sb, t + s) + \dot{u}(x + sb, t + s) = f(x + sb, t + s).$$

This function obeys

$$\begin{aligned} z(0) - z(-t) = u(x, t) - g(x - bt) &= \int_{-t}^{0} \dot{z}(s) ds \\ = \int_{-t}^{0} f(x + sb, t + s) ds &= \int_{0}^{t} f(x + (s - t)b, s) ds. \end{aligned}$$

Hence the solution u is equal to $\quad u(x, t) = g(x - bt) + \int_0^t f(x + (s - t)b, s) ds.$

We observe that this formula is analogous to the formula for solutions of inhomogenous initial value problems of linear ODE's. The unique solution is the sum of the unique solution of the corresponding homogenous initial value problem and the integral over solution of the homogenous equation with the inhomogenity as initial values.

Again one can show that the initial value problem has a unique solution in the sense of distributions. More precisely, for any function g which defines a distribution $F_g \in \mathcal{D}'(\mathbb{R}^n)$ and any function f which defines a distribution $F_f \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ there exists a unique distribution F, which solves the corresponding initial value problem of the inhomogenous transport equation. We obtained these solution of the first order homogenous and inhomogenous transport equation by solving an ODE. We now want to generalise this method and solve more general first order PDE's by solving an appropriate chosen system of first order ODE's.

2.3 Method of Characteristics

In this section we shall solve the following first order PDE:

$$F(\nabla u(x), u(x), x) = 0.$$

Here u is a real unknown function on an open domain $\Omega \subset \mathbb{R}^n$ and F is a real function on an open subset of $W \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$. We try to obtain the solution by solving an appropriate system of first order ODE's for the values of the function u along some integral curves along some vector fields. So let $x(s)$ be such such an integral curve and $p(s) = \nabla u(x(s))$ the gradient of the unknown function along this curve. We want to determine the curve $s \mapsto x(s)$ in such a way, that the tripple $s \mapsto (x(s), p(s), z(s))$ with $z(s) = u(x(s))$ solves an ODE. For this purpose we differentiate

$$\frac{dp_i(s)}{ds} = \frac{d}{ds}\frac{\partial u(x(s))}{\partial x_i} = \sum_{j=1}^n \frac{\partial^2 u(x(s))}{\partial x_j \partial x_i}\frac{dx_j(s)}{ds}.$$

The total derivative of $F(\nabla u(x), u(x), x) = 0$ with respect to x_i gives

$$\begin{aligned} 0 &= \frac{dF(\nabla u(x), u(x), x)}{dx_i} = \\ &= \sum_{j=1}^n \frac{\partial F(\nabla u(x), u(x), x)}{\partial p_j}\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \frac{\partial F(\nabla u(x), u(x), x)}{\partial z}\frac{\partial u(x)}{\partial x_i} + \frac{\partial F(\nabla u(x), u(x), x)}{\partial x_i}. \end{aligned}$$

Due to the commutativity $\partial_i \partial_j u = \partial_j \partial_i u$ of the second partial derivatives we obtain

$$\sum_{j=1}^n \frac{\partial F(p(s), z(s), x(s))}{\partial p_j}\frac{\partial^2 u(x(s))}{\partial x_j \partial x_i} + \frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s) + \frac{\partial F(p(s), z(s), x(s))}{\partial x_i} = 0.$$

Now we choose the vector field for the integral curves $s \mapsto x(s)$ as

$$\frac{dx_j}{ds} = \frac{\partial F(p(s), z(s), x(s))}{\partial p_j}.$$

This choice allows us to rewrite the differential equation

$$\frac{dp_i(s)}{ds} = \sum_{j=1}^n \frac{\partial^2 u(x(s))}{\partial x_j \partial x_i}\frac{dx_j}{ds}(s)$$

as

$$\frac{dp_i(s)}{ds} = \sum_{j=1}^{n} \frac{\partial^2 u(x(s))}{\partial x_j \partial x_i} \frac{\partial F(p(s), z(s), x(s))}{\partial p_j} =$$
$$= -\frac{\partial F(p(s), z(s), x(s))}{\partial x_i} - \frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s).$$

Finally we differentiate

$$\frac{dz(s)}{ds} = \frac{du(x(s))}{ds} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(x(s)) \frac{dx_j(s)}{ds} = \sum_{j=1}^{h} p_j(s) \frac{\partial F(p(s), z(s), x(s))}{\partial p_j}.$$

In this way we indeed obtain the following system of first order ODE's:

$$\begin{aligned}
\dot{x}_i(s) &= \frac{\partial F(p(s), z(s), x(s))}{\partial p_i} \\
\dot{p}_i(s) &= -\frac{\partial F(p(s), z(s), x(s))}{\partial x_i} - \frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s) \\
\dot{z}(s) &= \sum_{j=1}^{n} \frac{\partial F(p(s), z(s), x(s))}{\partial p_j} p_j(s).
\end{aligned}$$

This is a system of first order ODE's with $2n+1$ unknown real functions. Let us summarize these calculations in the following theorem:

Theorem 2.4. *Let F be a real differentiable function on an open subset $W \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ a twice differentiable solution on an open subset $\Omega \subset \mathbb{R}^n$ of the first order PDE $F(\nabla u(x), u(x), x) = 0$. For every solution $s \mapsto x(s)$ of the ODE*

$$\dot{x}_i(s) = \frac{\partial F}{\partial p_i}(\nabla u(x(s)), u(x(s)), x(s))$$

the functions $p(s) = \nabla u(x(s))$ and $z(s) = u(x(s))$ solve the ODE's

$$\dot{p}_i(s) = -\frac{\partial F(p(s), z(s), x(s))}{\partial x_i} - \frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s) \text{ and}$$
$$\dot{z}(s) = \sum_{j=1}^{n} \frac{\partial F(p(s), z(s), x(s))}{\partial p_j} p_j(s).$$

q.e.d.

Now we want to introduce a boundary value problem of the following form:

$$u(y) = g(y) \text{ for all } y \in \Omega \cap H \text{ with } H = \{y \in \mathbb{R}^n \mid y \cdot e_n = x_0 \cdot e_n\}.$$

Here $e_n = (0, \ldots, 0, 1)$ denotes the n–th element of the canocialcl basis and H the unique hyperplane through $x_0 \in \Omega$ ortogonal to e_n. By the implizit function theorem general continuously differentialble hypersurfaces may be brought into this form by a continuously differentiabel coordinate transformation with contiinuously differentiabel inverse: Let $\Phi : \Omega \to \Omega'$ ba a continuously differentiable homoemorphism with continuously differetiable inverse Φ^{-1}. Then the composition $u = v \circ \Phi$ of a function $v : \Omega' \to \mathbb{R}$ solves the PDE

$$F(\nabla u(x), u(x), x) = 0$$

if and only if v solves the PDE

$$F\left(\left(\Phi'\left(\Phi^{-1}(y)\right)\right)^t \cdot \nabla v(y), v(y), \Phi^{-1}(y)\right) = 0.$$

Therefore the PDE for the function v is the zero set of the function

$$G(\nabla v(y), v(y), y) = F\left(\left(\Phi'\left(\Phi^{-1}(y)\right)\right)^t \cdot \nabla v(y), v(y), \Phi^{-1}(y)\right)$$

In the sequel we assume that the hyperplane H has the following form:

$$H = \{y \in \mathbb{R}^n \mid y \cdot e_n = x_0 \cdot e_n\}.$$

If the hypersurface $H' \subset \Omega'$ is the zero set of a continuously differentiable function $\Lambda : \Omega' \to \mathbb{R}$ whose gradient $\nabla\Lambda$ does not vanish on H', then the implicit function theorem shows that in a neighbourhood of $y_0 \in H'$ there exists such a Φ. Furthermore, Φ is as often differentiable as Λ. In the foregoing theorem the functions u and v has to be twice differentiable. We assume that Φ and Φ^{-1} are twice differentiable. Consequently Λ should be twice differentiable. On $\Omega \cap H$ there must hold

$$F(\nabla u(y), u(y), y) = 0.$$

On order to define initial conditions at $y \in \Omega \cap H$

$$z(0) = g(y), \qquad p(0) = q(y) \quad \text{and} \qquad x(0) = y$$

we have to find a solution $q : \Omega \cap H \to \mathbb{R}^n, \quad y \mapsto q(y)$ of the following equation:

$$F(q(y), g(y), y) = 0 \quad \text{and} \qquad \frac{\partial g(y)}{\partial y_i} = q_i(y) \text{ for } i = 1, \ldots, n-1.$$

The second equations uniquely determine $q_1(y), \ldots, q_{n-1}(y)$ in terms of

$$\frac{\partial g(y)}{\partial y_1}, \ldots, \frac{\partial g(y)}{\partial y_{n-1}}.$$

It remains to determine the component $q_n(y)$ in such a way, that

$$F(q(y), g(y), y) = 0$$

holds for all $y \in \Omega \cap H$. Now the implicit function theorem implies the following lemma:

Lemma 2.5. *Let $F : W \to \mathbb{R}$ and $g : H \to \mathbb{R}$ be continuously differentiable, $x_0 \in \Omega \cap H$, $z_0 = g(x_0)$ and $p_{0,1} = \frac{\partial g(x_0)}{y_1}, \ldots, p_{0,n-1} = \frac{\partial g(x_0)}{y_n}$. If there exist $p_{0,n}$ with $(p_0, z_0, x_0) \in W$,*

$$F(p_0, z_0, x_0) = 0 \quad \text{and} \quad \frac{\partial F(p_0, z_0, x_0)}{\partial p_n} \neq 0,$$

then on an open neighbourhood of $x_0 \in \Omega \cap H$ there exists a unique solution q of

$$F(q(y), g(y), y) = 0, \quad q_i(y) = \frac{\partial g(y)}{\partial y_i} \text{ for } i = 1, \ldots, n-1 \quad \text{and} \quad q(y_0) = p_0. \quad \textbf{q.e.d.}$$

Theorem 2.6. *Let $F : X \to \mathbb{R}$ and $g : \Omega \cap H \to \mathbb{R}$ be three times differentiable functions on open subsets. Furthermore, let $(p_0, z_0, x_0) \in W$ and g statisfy $F(p_0, z_0, x_0) = 0$, $g(x_0) = z_0$, $p_{0,1} = \frac{\partial g(x_0)}{y_1}, \ldots, p_{0,n-1} = \frac{\partial g(x_0)}{y_n}$ and $\frac{\partial F}{\partial p_n}(p_0, z_0, x_0) \neq 0$. Then there exists on a neighbourhood Ω of x_0 a solution of the boundary value problem*

$$F(\nabla u(x), u(x), x) = 0 \quad \text{for} \quad x \in \Omega \quad \text{and} \quad u(y) = g(y) \quad \text{for} \quad y \in \Omega \cap H.$$

Proof. By the foregoing Lemma there exists a solution q on an open neighbourhood of x_0 in H of the following equations

$$F(q(y), g(y), y) = 0, \quad q_i(y) = \frac{\partial g(y)}{\partial y_i} \text{ for } i = 1, \ldots, n-1 \quad \text{and} \quad q(y_0) = p_0.$$

If F is twice and g are three times differentable then the implicit function theorem yields a twice differentiable solution. The theorem of Picard Lidenlöff shows that the following initial value problem has for all y in the intesection of an open neighbourhood of x_0 with H a unique solution:

$$\begin{aligned}
\dot{x}_i(s) &= \frac{\partial F}{\partial p_i}(p(s), z(s), x(s)) && \text{with} \quad x(0) = y \\
\dot{p}_i(s) &= -\frac{\partial F}{\partial x_i}(p(s), z(s), x(s)) - \frac{\partial F}{\partial z}(p(s), z(s), x(s))p_i(s) && \text{with} \quad p(0) = q(y) \\
\dot{z}(s) &= \sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(p(s), z(s), x(s))p_j(s) && \text{with} \quad z(0) = g(y).
\end{aligned}$$

We denote the family of solutions by $(x(y,s), p(y,s), z(y,s))$. For small $\Omega \ni x_0$ there exists an $\epsilon > 0$ such that these solutions are uniquely defined on $(y,s) \in (\Omega \cap H) \times (-\epsilon, \epsilon)$. Since F and g are three times differentiable all cofficients and initial values are twice differentiable. A theorem on the dependence of solutions of ODE's on the initial values yields that $(y,s) \mapsto (x(y,s), p(y,s), z(y,s))$ is on $(\Omega \cap H) \times (-\epsilon, \epsilon)$ twice differentiable. Due to the choice of the initial values at $s = 0$, the function

$$(\Omega \cap H) \times (-\epsilon, \epsilon) \to \Omega, \quad (y,s) \mapsto x(y,s)$$

has at $(y,s) = (x_0, 0)$ the Jacobi matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 & \frac{\partial F(p_0,z_0,x_0)}{\partial p_1} \\ & & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & \frac{\partial F(p_0,z_0,x_0)}{\partial p_{n-1}} \\ 0 & 0 & \dots & 0 & \frac{\partial F(p_0,z_0,x_0)}{\partial p_n} \end{pmatrix}.$$

Since $\frac{\partial F(p_0,z_0,x_0)}{\partial p_n} \neq 0$ this matrix is invertible, and the inverse function theorem implies that on an possibly diminished neighbourhood Ω of x_0 and an appropriate chosen $\epsilon > 0$ this map is a twice differentiable homoemorphims with twice differentiable inverse mapping. Now we define the function $u : \Omega \to \mathbb{R}$ by

$$u(x(y,s)) = z(y,s) \text{ for all } (y,s) \in (\Omega \cap H) \times (-\epsilon, \epsilon).$$

Our next task is to show that this function solves the PDE $F(\nabla u(x), u(x), x) = 0$.

In a first step we observe that the ODE implies

$$\frac{\partial}{\partial s} F(p(y,s), z(y,s), x(y,s)) = 0.$$

Since $F(q(y), g(y), y)$ vanishes for all $y \in \Omega \cap H$ we conclude

$$F(p(y,s), z(y,s), x(y,s)) = 0 \text{ for all } (y,s) \in (\Omega \cap H) \times (-\epsilon, \epsilon).$$

Hence it suffices to show $p(y,s) = \nabla u(x(y,s))$ for all $(y,s) \in (\Omega \cap H) \times (-\epsilon, \epsilon)$.

In a second step we show

$$\frac{\partial z(y,s)}{\partial s} = \sum_{j=1}^{n} p_j(y,s) \frac{\partial x_j(y,s)}{\partial s} \quad \text{and} \quad \frac{\partial z(y,s)}{\partial y_i} = \sum_{j=1}^{n} p_j(y,s) \frac{\partial x_j(y,s)}{\partial y_i}$$

for all $(y,s) \in (\Omega \cap H) \times (-\epsilon, \epsilon)$ and all $i = 1, \dots, n-1$. The first equation follows form the ODE for $x(y,s)$ and $z(y,s)$. For $s = 0$ the second equation follows from the initial conditions for $z(y,s)$, $p(y,s)$ and $x(y,s)$. The derivative of the first equation with respect to y_i yields

$$\frac{\partial^2 z(y,s)}{\partial y_i \partial s} = \sum_{j=1}^{n} \left(\frac{\partial p_j(y,s)}{\partial y_i} \frac{\partial x_j(y,s)}{\partial s} + p_j(y,s) \frac{\partial^2 x_j(y,s)}{\partial y_i \partial s} \right).$$

By the commutativity of the second partial derivatives we obtain

$$\begin{aligned}
&\frac{\partial}{\partial s} \left(\frac{\partial z(y,s)}{\partial y_i} - \sum_{j=1}^{n} p_j(y,s) \frac{\partial x_j(y,s)}{\partial y_i} \right) = \\
&\qquad = \sum_{j=1}^{n} \left(\frac{\partial p_j(y,s)}{\partial y_i} \frac{\partial x_j(y,s)}{\partial s} - \frac{\partial p_j(y,s)}{\partial s} \frac{\partial x_j(y,s)}{\partial y_i} \right) = \\
&\qquad = \sum_{j=1}^{n} \frac{\partial p_j(y,s)}{\partial y_i} \frac{\partial F(p(y,s),z(y,s),x(y,s))}{\partial p_j} + \\
&+ \sum_{j=1}^{n} \left(\frac{\partial F(p(y,s),z(y,s),x(y,s))}{\partial x_j} + \frac{\partial F(p(y,s),z(y,s),x(y,s)) p_j(y,s)}{\partial z} \right) \frac{\partial x_j(y,s)}{\partial y_i} \\
&\qquad = \frac{\partial}{\partial y_i} F(p(y,s),z(y,s),x(y,s)) - \\
&\qquad - \frac{\partial F(p(y,s),z(y,s),x(y,s))}{\partial z} \left(\frac{\partial z(y,s)}{\partial y_i} - \sum_{j=1}^{n} p_j(y,s) \frac{\partial x_j(y,s)}{\partial y_i} \right).
\end{aligned}$$

We insert the result $F(p(y,s),z(y,s),x(y,s)) = 0$ of the first step and obtain

$$\begin{aligned}
&\frac{\partial}{\partial s} \left(\frac{\partial z}{\partial y_i}(y,s) - \sum_{j=1}^{n} p_j(y,s) \frac{\partial x_j(y,s)}{\partial y_i} \right) = \\
&\qquad = -\frac{\partial F(p(y,s),z(y,s),x(y,s))}{\partial z} \left(\frac{\partial z}{\partial y_i}(y,s) - \sum_{j=1}^{n} p_j(y,s) \frac{\partial x_j(y,s)}{\partial y_i} \right).
\end{aligned}$$

This is a linear homogenous ODE with initial value 0 at $s = 0$. The unique solution vanishes identically. This implies the second equation and finishes the second step:

$$\frac{\partial z(y,s)}{\partial y_i} = \sum_{j=1}^{n} p_j(y,s) \frac{\partial x_j(y,s)}{\partial y_i}.$$

Finally in a third step we show $p(y,s) = \nabla u(x(y,s))$ for all $(y,s) \in (\Omega \cap H) \times (-\epsilon, \epsilon)$. Locally the derivative of the map $(y,s) \mapsto x$ is invertible. Alltogether we obtain

$$\begin{aligned}\frac{\partial u}{\partial x_j} = \frac{\partial z}{\partial s}\frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1} \frac{\partial z}{\partial y_i}\frac{\partial y_i}{\partial x_j} &= \left(\sum_{k=1}^{n} p_k \frac{\partial x_k}{\partial s}\right)\frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1}\left(\sum_{k=1}^{n} p_k \frac{\partial x_k}{\partial y_i}\right)\frac{\partial y_i}{\partial x_j} \\ = \sum_{k=1}^{n} p_k \left(\frac{\partial x_k}{\partial s}\frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1}\frac{\partial x_k}{\partial y_i}\frac{\partial y_i}{\partial x_j}\right) &= \sum_{k=1}^{n} p_k \frac{\partial x_k}{\partial x_j} = p_j.\end{aligned}$$

Due to the initial values $z(y,0)$ we have $u(y) = g(y)$ for all $y \in \Omega \cap H$. Die uniqueness of the solution follows from the Theorem 2.4 and the theorem of Picard-Lindelöf. **q.e.d.**

We solved the boundary value problem by solving a family of ODE's. In the case of the inhomogenous transport equation we combine the coordinates x and t to one coordinate (x,t). Consequently we write

$$F(p, z, (x,t)) = \tilde{F}(p, x, t) = b_1 p_1 + \ldots + b_n p_n + p_{n+1} - f(x,t).$$

We use the equation $F(p, z, (x,t) = 0$ and rewrite the ODE for z. Then the ODE bcomes independent of p and we can solve $x(s)$, $t(s)$ and $z(s)$ separately:

$$\dot{x} = b \qquad \dot{t} = 1 \qquad \dot{p} = (\nabla f(x,t), \dot{f}(x,t)) \qquad \dot{z} = \tilde{F}(p,x,t) + f(x,t) = f(x,t).$$

Whenever the function F depends linearly on p, then the functions

$$\frac{\partial F(p(s), z(s), x(s))}{\partial p_i} \text{ for } i = 1, \ldots, n \text{ and}$$
$$F(p(s), z(s), x(s)) - \sum_{j=1}^{n} \frac{\partial F(p(s), z(s), x(s))}{\partial p_j} p_j(s)$$

do not depend on p. Therefore the ODE system becomes independent of $p(s)$, and the components $x(s)$ and $z(s)$ can be solved independently of $p(s)$. This situation describes the transport equation with vector b depending on x and t. For the solution of this equation we do not need to introduce the function $p(s) = \nabla u(x(s))$.