Chapter 6

Scalar Conservation Laws

6.1 Crossing Characteristics

In this section we consider the non-linear first order differential equation

$$\dot{u}(x,t) + \nabla f(u(x,t)) = \dot{u}(x,t) + f'(u(x,t)) \cdot \nabla u(x,t) = 0$$

with a smooth function $f : \mathbb{R} \to \mathbb{R}^n$. Here $u : \mathbb{R}^n \times \mathbb{R}$ is the unknown function. We impose the initial conditions $u(x,t) = u_0(x)$ with some given function $u_0 : \mathbb{R}^n \to \mathbb{R}$. For any bounded open subset $\Omega \subset \mathbb{R}^n$ for which the divergence theorem holds we conclude

$$\frac{d}{dt} \int\limits_{\Omega} u(x,t) d^n x = \int\limits_{\Omega} \dot{u}(x,t) d^n x = -\int\limits_{\Omega} \nabla f(u(x,t)) d^n x = -\int\limits_{\partial\Omega} f(u(x,t)) \cdot N(x) d\sigma(x).$$

This is the meaning to be a conservation law: the change of the integral over $u(x,t)$ is equal to the flux of $f(u(x,t))$ through the boundary $\partial\Omega$.

This equation is a non-linear first order PDE, and we apply the method of characteristic. With $(p_1, \ldots, p_n) = \nabla u$ and $p_{n+1} = \dot{u}$ and $z = u$ the PDE takes the form $p_{n+1} + f'(z) \cdot (p_1, \ldots, p_n) = 0$. It depends linearly on p and we may neglect p:

$$x'(s) = f'(u(x,t)), \quad t'(s) = 1, \quad z'(s) = f'(u) \cdot (p_1, \ldots, p_n) + p_{n+1} = 0.$$

This implies $s = t$ and $u(x(t), t) = u_0(x_0)$ along the solutions. Consequently the characteristic equation has for all $x_0 \in \mathbb{R}^n$ the solution

$$x(t) = x_0 + tf'(u_0(x_0)) \quad \text{and} \quad u(x_0 + tf'(u_0(x_0)), t) = u_0(x_0).$$

The solutions for initial values $x_1, x_2 \in \mathbb{R}^n$ with $u_0(x_1) \neq u_0(x_2)$ might intersect at $t \in \mathbb{R}^+$. In this case the method of characteristic implies $u_0(x_1) = u(x_1 + tf'(u_0(x_1)), t) =$

$u(x_2+tf'(u_0(x_2)),t) = u_0(x_2)$, which is impossible. This intersection of solutions of the characteristic equations is called crossing characteristics. For $n = 1$ there is a crossing of characteristics for $f'(u_0(x_2)) < f'(u_0(x_1))$ with $x_2 > x_1$.

Theorem 6.1. *For $f \in C^2(\mathbb{R},\mathbb{R})$ and $u_0 \in C^1(\mathbb{R},\mathbb{R})$ with $f''(u_0(x))u_0'(x) > -\alpha$ for all $x \in \mathbb{R}$ and some $\alpha \geq 0$ there is a unique C^1-solution of the initial value problem*

$$\frac{\partial u(x,t)}{\partial t} + f'(u(x,t))\frac{\partial u(x,t)}{\partial x} = 0 \quad \textit{with} \quad u(x,0) = u_0(x)$$

on $(x,t) \in \mathbb{R} \times [0,\alpha^{-1})$ for $\alpha > 0$ and on $(x,t) \in \mathbb{R} \times [0,\infty)$ for $\alpha = 0$.

Proof. By the method of characteristic the solution $u(x,t)$ is on the lines $x+tf'(u_0(x))$ equal to $u_0(x)$. For all $t \geq 0$ with $1+t\alpha > 0$ the derivative of $x \mapsto x+tf'(u_0(x))$ obeys

$$1 + tf''(u_0(x))u_0'(x) \geq 1 + t\alpha > 0.$$

This implies that this map is C^1-diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. Therefore there exists for all $y \in \mathbb{R}$ a unique x with $x + tf'(u_0(x)) = y$. Then $u(y,t) = u_0(x)$ solves the inital value problem. **q.e.d.**

Example 6.2. *For $n = 1$ and $f(u) = \frac{1}{2}u^2$ we obtain Burgers equation:*

$$\dot{u}(x,t) + u(x,t)\frac{\partial u(x,t)}{\partial x} = 0.$$

The solutions of the corresponding characteristic equations are $x(t) = x_0 + u_0(x_0)t$. Therefore the solutions of the corresponding initial value problem obey

$$u(x + tu_0(x),t) = u_0(x).$$

If u_0 is continuously differentialble and monotonic increasing, then for all $t \in [0,\infty)$ the map $x \mapsto x + tu_0(x)$ is a C^1-diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ and there is a unique C^1-solution on $\mathbb{R} \times [0,\infty)$. More generally, if $u_0'(x) > -\alpha$ with $\alpha \geq 0$, then there is a unique C^1-solution on $\mathbb{R} \times [0,\alpha^{-1})$ for $\alpha > 0$ and $(x,t) \in \mathbb{R} \times [0,\infty)$ for $\alpha = 0$.

6.2 Admissible Solutions

In this section we look for more general notions of solutions which allows to extends solutions across the crossing characteristics. For this purpose we use the preserved integrals. Since we restrict to the one-dimensional situation the natural domians are intervalls $\Omega = [a, b]$ with $a < b \in \mathbb{R}$. In this case the conservation law implies

$$\frac{d}{dt}\int\limits_a^b u(x,t)dx = f(u(a,t)) - f(u(b,t)).$$

Now we look for functions u with discontinuities along the graph $\{(x,t) \mid x = y(t)\}$ of a C^1–function y. In case $y(t)$ belongs to $[a, b]$ we split the integral over $[a, b]$ into the integrals over $[a, b] = [a, y(t)] \cup [y(t), b]$. Let us now calculate in this case the derivative of the integral over $[a, b]$:

$$\frac{d}{dt}\int\limits_a^b (u(x,t)dx = \frac{d}{dt}\int\limits_a^{y(t)} u(x,t)dt + \frac{d}{dt}\int\limits_{y(t)}^b u(x,t)dx =$$

$$= \dot{y}(t)\lim_{x\uparrow y(t)} u(x,t) + \int\limits_a^{y(t)} \dot{u}(x,t)dx - \dot{y}(t)\lim_{x\downarrow y(t)} u(x,t) + \int\limits_{y(t)}^b \dot{u}(x,t)dx.$$

We abbreviate $\lim_{x\uparrow y(t)} u(x,t)$ as $u^l(y(t),t)$ and $\lim_{x\downarrow y(t)} u(x,t)$ as $u^r(y(t),t)$ and assume that on both sides of the graph of y the function u is a classical solution of the conservation law:

$$\frac{d}{dt}\int\limits_a^b (u(x,t)dx = \dot{y}(t)(u^l(y(t),t) - u^r(y(t),t)) - \int\limits_a^{y(t)} \frac{d}{dx}f(u(x,t))dx - \int\limits_{y(t)}^b \frac{d}{dx}f(u(x,t))dx$$

$$= \dot{y}(t)(u^l(y(t),t) - u^r(y(t),t)) + f(u(a,t)) - f(u(b,t)) + f(u^r(y(t),t) - f(u^l(y(t),t).$$

Hence the integrated version of the conservation law still holds, if the following Rankine-Hugonoit condition is fullfilled:

$$\dot{y}(t) = \frac{f(u^r(y,t)) - f(u^l(y,t)}{u^r(y,t) - u^l(y,t)}.$$

Example 6.3. *We consider Burgers equation* $\dot{u}(x,t) + u(x,t)\frac{\partial u}{\partial x}(x,t) = 0$ *for* $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ *with the following continuous initial values* $u(x,0) = u_0(x)$ *and*

$$u_0(x) = \begin{cases} 1 & \text{for } x \le 0, \\ 1 - x & \text{for } 0 \le x < 1 \\ 0 & \text{for } 1 \le x. \end{cases}$$

The first crossing of characterstics happens for $t = 1$:

$$x + tu_0(x) = \begin{cases} x + t & \text{for } x \leq 0, \\ x + t(1 - x) & \text{for } 0 < x < 1, \\ x & \text{for } 1 \leq x. \end{cases}$$

For $t < 1$ the evaluation at t is a homoemorphism from $\mathbb{R}$ onto itself with inverse

$$x \mapsto \begin{cases} x - t & \text{for } x \leq t, \\ \frac{x-t}{1-t} & \text{for } t < x < 1, \\ x & \text{for } 1 \leq x. \end{cases}$$

Therefor the solution is for $1 < t$ equal to

$$u(x,t) = \begin{cases} 1 & \text{for } x \leq t, \\ \frac{x-1}{t-1} & \text{for } t \leq x < 1, \\ 0 & \text{for } 1 \leq x. \end{cases}$$

At $t = 1$ the solutions of the characteristic equations starting at $x \in [0,1]$ all meet at $x = 1$. For $t > 1$ there exists a unique discontinuous solution satisfying the Rankine-Hugonoit condition. For small x this solution is 1 and for large x it is 0. The corresponding regions has to be separated by a path with velocity $-\frac{1}{2}$ which starts at $(x,t) = (1, 1$. For $t \geq 1$ this discontinuous solution is equal to

$$u(x,t) + \begin{cases} 1 & \text{for } x \leq 1 + \frac{t}{2}, \\ 0 & \text{for } 1 + \frac{t}{2} < x. \end{cases}$$

The second initial value problem is not continous but monotonic increasing. For continous monotonic increasing functions u_0 the evaluation at t of the solutions of the characteristic equation would be a homoemorphism for all $t > 0$. Therefore in such cases there exists a unique continuous solution for all $t > 0$. But for non-continuous initial values this is not the case.

Example 6.4. *We again consider Burgers equation $\dot{u}(x,t) + u(x,t)\frac{\partial u}{\partial x}(x,t) = 0$ for $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ with the following non-continuous initial values $u(x,0) = u_0(x)$ and*

$$u_0(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } 0 < x. \end{cases}$$

Again there is a unique discontinuous solution which is for small x equal to 0 and for large x equal to 1. By the Rankine-Hugonoit condition both regions are separated by a path with velocity $\frac{1}{2}$. This soluiton is equal to

$$u(x,t) = \begin{cases} 0 & \textit{for } x < \frac{t}{2}, \\ 1 & \textit{for } \frac{1}{2} \leq x. \end{cases}$$

But there exists another continuous solution, which clearly also satisfies the Rankine-Hugonoit condition:

$$u(x,t) = \begin{cases} 0 & \textit{for } x \leq 0, \\ \frac{x}{t} & \textit{for } 0 \leq x \leq t, \\ 1 & \textit{for } 1 < x. \end{cases}$$

Besides these two extreme cases there exists infinitley many other solutions with several regions of discontinuity, which all satisfy the Rankine-Hugonoit condition.

These examples show that such weak solutions exists for all $t \geq 0$ but are not unique. Therefore we want to restrict the space of weak solutions such that they have a unique soltions for all $t \geq 0$. Since we want to maximise the regularity we only accept discontinuouties, if there are no continuous solutions. In the last example we prefer the continuous solution. So for Burgers equation this means we only accept discontinuous solutions, which take larger values for smaller x and smaller values for larger x.

Definition 6.5 (Lax Entropy condition)**.** *A discontinuity of a weak solution along a C^1–path $t \mapsto y(t)$ satisfies the Lax entropy condition, if along the path the following inequality is fullfilled:*

$$f'(u^l(y,t),t) > \dot{y}(t) > f'(u^r(y,t)).$$

A weak solutions with discontinuities along C^1–paths is called an admissible solution, if along the path both the Rankine–Hugonoit condition and the Lax Entropy condition is satisfied.

For continuous u_0 there is a crossing of characteristics if $f'(u_0(x_1)) > f'(u_0(x_2))$ for $x_1 < x_2$. So this condition ensures that discontinuities can only show up if we cannot avoid a crossing of characteristics.

Theorem 6.6. *Let $f \in C^1(\mathbb{R},\mathbb{R})$ be convex and u and v two admissible solutions of*

$$\dot{u}(x,t) + f'(u(x,t))\frac{\partial u}{\partial x}(x,t) = 0.$$

in $L^1(\mathbb{R})$. Then $t \mapsto \|u(\cdot,t) - v(\cdot;t)\|_{L^1\mathbb{R})}$ is monotonically decreasing.

Proof. We divide $\mathbb{R}$ into maximal intervalls $I = [a(t), b(t)]$ with the property that either $u(x,t) > v(x,t)$ or $v(x,t) > u(x,t)$ for all $x \in (a(t), b(t))$. This means that either $x \mapsto u(x,t) - v(x,t)$ vanishes at the boundary, or is discontinuous and changes sign at the boundary. We claim that the boundaries $a(t)$ and $b(t)$ of these maximal intervalls are differentiable. We prove this only for $a(t)$. For $b(t)$ the proof is analogous. If either $u(\cdot,t)$ or $v(\cdot,t)$ is discontinuous at $a(t)$, then by definition of an admissible solution the locus of the discontinuity $a(t)$ is differentiable with respect to t. If $u(\cdot,t)$ and $v(\cdot,t)$ are both continuously differentiable at $a(t)$ with $u(a,t) = v(a,t)$, then by the method of characteristic for sufficiently small $\epsilon > 0$ all $x \in (a(t)-\epsilon, a(t)+\epsilon)$ with $u(x,t) = v(x,t)$ preserve this property along the solutions of $\dot{x}(t) = f'(u(x(t),t) = f'(v((x(t),t)$. This implies that $a(t)$ is differentiable with $\dot{a}(t) = f(u(a,t)) = f(v(a,t))$. Let us only consider intervalls on whose interior $u(\cdot,t) - v(\cdot,t)$ is positive. For the other intervalls we apply the same arguments with interchanged u and v. Now we calculate

$$
\begin{aligned}
\frac{d}{dt}\int\limits_{a(t)}^{b(t)} (u(x,t) - v(x,t))dx &= \int\limits_{a(t)}^{b(t)} (\dot{u}(x,t) - \dot{v}(x,t))dx + \\
&+ \dot{b}(t)(u(b(t),t) - v(b(t),t)) - \dot{a}(t)(u((a,t) - v(a,t)) \\
&= \int\limits_{a(t)}^{b(t)} \frac{d}{dx}(f(v(x,t) - f(u(x,t))dx \\
&+ \dot{b}(t)(u(b(t),t) - v(b(t),t)) - \dot{a}(t)(u((a,t) - v(a,t)) \\
&= f(v(b(t),t) - f(u(b(t),t) + \dot{b}(t)(u(b(t),t) - v(b(t),t)) \\
&+ f(u(a,t) - f(v(a,t) + \dot{a}(t)(v(a,t) - u(a,t)).
\end{aligned}
$$

If $u(\cdot,t)$ and $v(\cdot,t)$ are both differentiable at $a(t)$, then they take the same values at $a(t)$ and the last line vanishes. Analogously, if $u(\cdot,t)$ and $v(\cdot,t)$ are both differentiable at $b(t)$, then the second last line vanishes. For convex f the derivative f' is monotonically increasing and the Lax-Entropy condition implies

$$
u^l(y,t) > u^r(y,t), \qquad\qquad v^l(y,t) > v^r(y,t)
$$

at all discontinuities y of $u(\cdot,t)$ and $v(\cdot,t)$, respectively. If one of the two solutions u and v is at the boundary of I continuous and the other is non-continuous, then by definition of the intervall I the value of the continuous solution has to lie in bewteen the the limits of the non-continuous solution. Therefore either $u(\cdot,t)$ is continuous and diffierentiabel at $a(t)$ and $v(\cdot,t)$ is discontinuous at $a(t)$ or u is discontinuous at $b(t)$ and v is continuous and differentiable at $b(t)$. In the first case we use the Rankine

Hugonoit condition to determine $\dot{a}(t)$ and $\dot{b}(t)$. The corresponding contribution to the derivative of $\|u(\cdot,t)-v(\cdot,t)\|_1$ is

$$\begin{aligned}
&f(u(a,t)) - f(v^r(a,t)) + \dot{a}(t)\,(v^r(a,t)-u(a,t)) = \\
&\quad = f(u(a,t)) - f(v^r(a,t)) + \frac{f(v^r(a,t))-f(v^l(a,t))}{v^r(a,t)-v^l(a,t)}\,(v^r(a,t)-u(a,t)) \\
&\quad = f(u(a,t)) - \left(f(v^r(a,t))\frac{v^l(a,t)-u(a,t)}{v^l(a,t)-v^r(a,t)} + f(v^l(a,t))\frac{u(a,t)-v^r(a,t)}{v^l(a,t)-v^r(a,t)}\right).
\end{aligned}$$

Since f is convex the secant lies above the graph of f. Hence due to $u(a,t) \in [v^r(a,t), v^l(a,t)]$ this expression is non-positive. In the second case the contribution to the derivative of $\|u(\cdot,t)-v(\cdot,t)\|_1$ is

$$\begin{aligned}
&f(v(b,t)) - f(u^l(b,t)) + \dot{b}(t)\,\big(u^l(b,t)-v(b,t)\big) = \\
&\quad = f(v(b,t)) - f(u^l(b,t)) + \frac{f(u^r(b,t))-f(u^l(b,t))}{u^r(b,t)-u^l(b,t)}\,\big(u^l(b,t)-v(b,t)\big) \\
&\quad = f(v(b,t)) - \left(f(u^r(b,t))\frac{u^l(b,t)-v(b,t)}{u^l(b,t)-u^r(b,t)} + f(u^l(b,t))\frac{v(b,t)-u^r(b,t)}{u^l(b,t)-u^r(b,t)}\right).
\end{aligned}$$

Again due to $v(b,t) \in [u^r(b,t), u^l(b,t)]$ this expression is non-positive.

If finally both solutions are discontinous at $a(t)$ or $b(t)$. Since $u(\cdot,t)-v(\cdot,t)$ is positive on I, the Lax Entropy condition implies $u^r(a,t) \in [v^l(a,t), v^r(a,t)]$ and $v^l(b,t) \in [u^l(b,t), u^r(b,t)]$, respectively. The corresponding contributions to the derivative of $\|u(\cdot,t)-v(\cdot,t)\|_1$ are again non-positve:

$$\begin{aligned}
&f(u^r(a,t)) - f(v^r(a,t)) + \dot{a}(t)\,(v^r(a,t)-u^r(a,t)) = \\
&\quad = f(u^r(a,t)) - f(v^r(a,t)) + \frac{f(v^r(a,t))-f(v^l(a,t))}{v^r(a,t)-v^l(a,t)}\,(v^r(a,t)-u^r(a,t)) \\
&\quad = f(u^r(a,t)) - \left(f(v^r(a,t))\frac{v^l(a,t)-u^r(a,t)}{v^l(a,t)-v^r(a,t)} + f(v^l(a,t))\frac{u^r(a,t)-v^r(a,t)}{v^l(a,t)-v^r(a,t)}\right).
\end{aligned}$$

$$\begin{aligned}
&f(v^l(b,t)) - f(u^l(b,t)) + \dot{b}(t)\,\big(u^l(b,t)-v^l(b,t)\big) = \\
&\quad = f(v^l(b,t)) - f(u^l(b,t)) + \frac{f(u^r(b,t))-f(u^l(b,t))}{u^r(b,t)-u^l(b,t)}\,\big(u^l(b,t)-v^l(b,t)\big) \\
&\quad = f(v^l(b,t)) - \left(f(u^r(b,t))\frac{u^l(b,t)-v^l(b,t)}{u^l(b,t)-u^r(b,t)} + f(u^l(b,t))\frac{v^l(b,t)-u^r(b,t)}{u^l(b,t)-u^r(b,t)}\right).
\end{aligned}$$

This shows that the contributions to the derivative $\|u(\cdot,t)-v(\cdot,t)\|_1$ of all intervalls are non-positive. **q.e.d.**

This implies that admissible solutions are unique, if they exist. Let us now turn the question of theire existence. Let $u(x,t)$ be a solution of the conservation law

$$\dot{u}(x,t) + \frac{\partial}{\partial x} f(u(x,t)) = \dot{u}(x,t) + f'(u(x,t))\frac{\partial u(x,t)}{\partial x} = 0$$

such that $u(\cdot,t)$ belongs to $L^1(\mathbb{R})$ for all $t \geq 1$. This differential equation invlolves only the derivative of the function f. We asume $f(0) = 0$. Then the function

$$U(x,t) = \int_{-\infty}^{x} u(y,t)dy.$$

solves the differential equation

$$\dot{U}(x,t) + f(u(x,t)) - \lim_{x\to-\infty} f(u(x,t) = \dot{U}(x,t) + f\left(\frac{\partial U(x,t)}{\partial x}\right) = 0.$$

Now we assume that $f \in C^2(\mathbb{R},\mathbb{R})$ is strictly convex:

$$f(u) \geq f(v) + f'(v)(u-v) \quad \text{for all } u, v \in \mathbb{R}.$$

For $u = u(x,t)$ and $v \in \mathbb{R}$ obtain

$$-\dot{U}(x,t) = f(u(x,t)) \geq f(v) + f'(v)(u(x,t) - v).$$

We rewite this inequality as

$$\dot{U}(x,t) + f'(v)\frac{\partial U(x,t)}{\partial x} \leq g(v) \quad \text{with } g(v) = f'(v)v - f(v).$$

The left hand sinde is the derivative $\frac{d}{dt}U(x(t),t)$ along the curve with constant speed $f'(v)$ through the point (x,t). this curve is given by $x(t) = y + f'(v)t$, where y is the postion of this curve at $t = 0$. So we may rewright the forgoing inequality as

$$\frac{d}{dt}U(y + f'(v)t, t) = \dot{U}(y + f(v)t, t) + f'(v)\frac{\partial U(y + f'(v)t, t)}{\partial x} \leq g(v).$$

The integral from $t = 0$ to t gives

$$U(x,t) = U(y + f(v), t) \leq U(y,0) + tg(v).$$

Now we assume in addition that the strictly monotonic increasing function f' is a bijective function form $\mathbb{R}$ to $\mathbb{R}$. Let b denote the inverse function. For fixed (x,t) the relation $y + f'(v)t = x$ results in the folowing bijective relation between v and y:

$$y = x - tf'(v) \quad \iff \quad f'(v) = \frac{x-y}{t} \quad \iff \quad v = b\left(\frac{x-y}{t}\right).$$

Since b is the inverse function of f' theire derivatives obey $f''(b(s))b'(s) = 1$. Therefore $g'(v) = f''(v)v$ gives the following derivative of the function $h(s) = g(b(s))$:

$$h'(s) = g'(b(s))b'(s) = f''(b(s))b(s)b'(s) = b(s).$$

The condition $f(0) = 0$ implies that $g(0) = 0$ and the first inequality for $u = 0$ shows that this is the unique minimum of g. For $c = f'(0)$ we have $b(c) = 0$ and $h(c) = g(b(c)) = g(0) = 0$ and this is the unique minimum of h. Since f' strictly monotonic increasing the inverse function b is also strictly monotonic increasing. So we may rewrite the forgoing inquality in terms of h as

$$U(x,t) \leq U(y,0) + th\left(\frac{x-y}{t}\right).$$

In fact, since the forgoing inquality holds for all $v \in \mathbb{R}$ this inquality holds for all $y \in \mathbb{R}$. In the forgoing inequalities we have equality only for $v = u(x,t)$. Therefore there exists a unique y with equality in the last inquality two. This proves the following Theorem:

Theorem 6.7. *Let $f \in C^2(\mathbb{R}, \mathbb{R})$ be strictly convex, $f(0) = 0$ and $f' : \mathbb{R} \to \mathbb{R}$ bijective. Then any C^1-solution $u(x,t)$ of the initial value problem*

$$\dot{u}(x,t) + f'(u(x,t))\frac{\partial u(x,t)}{\partial x} = 0 \qquad \textit{with} \qquad u(x,0) = u_0(x)$$

is given by

$$u(x,t) = b\left(\frac{x - y(x,t)}{t}\right),$$

where $y(x,t)$ minimizes for all (x,t) the function

$$\int_{-\infty}^{y} u_0(z)dz + th\left(\frac{x-y}{t}\right) = U_0(y) + th\left(\frac{x-y}{t}\right) = G(x,y,t).$$

Here b is the inverse function of f' and h is the anti–derivative of b with $h(f'(0)) = 0$.

Let us now generalise this construction of a weak solution. We assume that $f \in C^2(\mathbb{R}, \mathbb{R})$ is strictly convex, $f(0) = 0$ and $f' : \mathbb{R} \to \mathbb{R}$ is bijective. Again we obtain

$$f(u) \geq f(v) + f'(v)(u - v) \quad \text{for all } u, v \in \mathbb{R}$$

with equality only for $v = u$. this implies

$$g(v) = f'(v) - f(v) \geq f'(v)u - f(u) \quad \text{for all } u, v \in \mathbb{R}$$

with equality only for $v = u$. For $u = 0$ we get $g(v) \geq 0$ with equality only for $v = 0$. If b is the inverse function of f', then the function $h(s) = g(b(s))$ is non-negative and has only one zero at $s = c = f'(0)$ and is strictly convex with $h'(s) = b(s)$. The function

$$U_0(y) = \int\limits_{-\infty}^{y} u(x)dx \quad \text{with } u_0 \in L^1(\mathbb{R})$$

is continuous and bounded. Therefore for fixed $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ the continuous function

$$G(x,y,t) = U_0(y) + th\left(\frac{x-y}{t}\right)$$

is bounded from below with the limits $\lim_{y \to \pm\infty} G(x,y,t) = \infty$. We define

$$V_n(x,t) = \ln\left(\int\limits_{-\infty}^{\infty} e^{-nG(x,y,t)}dy\right).$$

The conditions on G imply that the integral converges and is always positive. By definition of $g(v) = f'(v)v - f(v)$ and $h(s) = g(b(s))$ we have

$$h(s) - sb(s) = g(b(s)) - sb(s) = f'(b(s))b(s) - f(b(s)) - sb(s) = -f(b(s)).$$

Hence the functions G and V_n have the derivatives

$$\begin{aligned}
\frac{\partial G(x,y,t)}{\partial x} &= b\left(\frac{x-y}{t}\right), \\
\frac{\partial G(x,y,t)}{\partial t} &= h\left(\frac{x-y}{t}\right) - \frac{x-y}{t}b\left(\frac{x-y}{t}\right) = -f\left(b\left(\frac{x-y}{t}\right)\right), \\
u_n(x,t) &= -\frac{1}{n}\frac{\partial V_n}{\partial x} = \frac{\int_{-\infty}^{\infty} b\left(\frac{x-y}{t}\right)e^{-nG(x,y,t)}dy}{\int_{-\infty}^{\infty} e^{-nG(x,y,t)}dy}, \\
f_n(x,t) &= \frac{1}{n}\frac{\partial V_n}{\partial t} = \frac{\int_{-\infty}^{\infty} f\left(b\left(\frac{x-y}{t}\right)\right)e^{-nG(x,y,t)}dy}{\int_{-\infty}^{\infty} e^{-nG(x,y,t)}dy}.
\end{aligned}$$

Therefore the seqeunce of functions $u_n(x,t)$ und $f_n(x,t)$ solve the conservation law

$$\dot{u}_n(x,t) + \frac{\partial}{\partial x}f_n(x,t) = 0.$$

Furthermore, under the conditions of the theorem the sequences $u_n(x,t)$ and $f_n(x,t)$ converge in the limit $n \to \infty$ to the solution $u(x,t)$ and to $f(u(x,t))$, respectively. It can be shown, that u_n converges in $L^1(\mathbb{R})$ to the unique admissible solution.