Analysis II - Übung 11

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Aufgabe 2

$R := \{(x,y) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ \mid x-1 < y < x\}$
$S := \{(x,y) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ \mid x-2 < y < x-1\}$

(a) Behauptung

$R(y) := \{x \in \mathbb{R} \mid (x,y) \in R\}$ und
$S(y) := \{x \in \mathbb{R} \mid (x,y) \in S\}$ sind Intervalle

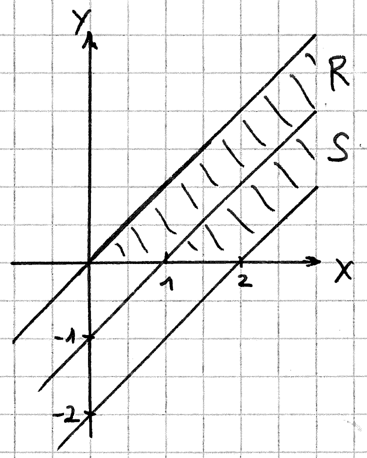
Beweis

Wir halten y fest.

$R = \{(x,y) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ \mid x-1 < y < x\}$
$\Rightarrow \quad x-1 < y \quad \wedge \quad y < x$
$\Rightarrow \quad x < y+1 \qquad \Rightarrow \quad y < x < y+1 \Rightarrow \underbrace{R(y) = (y, y+1)}_{\text{Intervall}}$

$S = \{(x,y) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ \mid x-2 < y < x-1\}$
$\Rightarrow \quad x-2 < y \quad \wedge \quad y < x-1$
$\Rightarrow \quad x < y+2 \quad \wedge \quad y+1 < x \Rightarrow \quad y+1 < x < y+2 \Rightarrow \underbrace{S(y) = (y+1, y+2)}_{\text{Intervall}}$

Skizze:

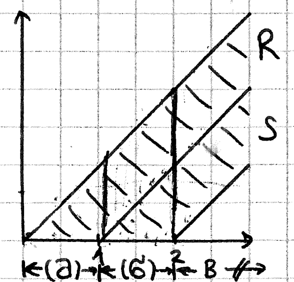


(b) **Behauptung**

Für $f = \chi_S - \chi_R$ ist $\int_{\mathbb{R}^2} f(x,y)\,dx\,dy \neq \int_{\mathbb{R}^2} f(x,y)\,dy\,dx$

Beweis

Nocheinmal die Skizze:



$$\int_{\mathbb{R}^2} f(x,y)\,dx\,dy = \int_{\mathbb{R}^2} \chi_S - \chi_R\,dx\,dy = \int_{\mathbb{R}}\int_{\mathbb{R}} \chi_S - \chi_R\,d\mu(x)\,d\mu(y)$$

$$= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} \chi_S\,d\mu(x) - \int_{\mathbb{R}} \chi_R\,d\mu(x)\right)d\mu(y)$$

$$= \int_{\mathbb{R}} \underbrace{\left(\int_{y+1}^{y+2} 1\,dx - \int_{y}^{y+1} 1\,dx\right)}_{y+2-(y+1)-(y+1-y)=0} d\mu(y) = \boxed{0}$$

$$\int_{\mathbb{R}^2} f(x,y)\,dy\,dx = \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_S - \chi_R\,d\mu(y)\,d\mu(x)$$

(a) $\forall\ 0 \leq x \leq 1$ (siehe Skizze)

$$\int -\chi_R\,d\mu(y) = \int_0^x -1\,dy = -x$$

(b) $\forall\ 1 \leq x \leq 2$

$$\int \chi_S - \chi_R\,d\mu(y) = \int_0^{x-1} 1\,dy - \int_{x-1}^{x} 1\,dy$$

$$= x-1-(x-(x-1))$$
$$= x-2$$

(B) $\forall\ x \geq 2$

$$\int \chi_S - \chi_R\,d\mu(y) = \int_{x-2}^{x-1} 1\,dy - \int_{x-1}^{x} 1\,dy$$

$$\left[= x-1-(x-2)-(x-(x-1))\right]$$
$$= 0$$

Der Satz von Fubini ist nicht verletzt, weil $f \notin L^1(\mathbb{R}^d)$! Die Integrale von R und S sind jeweils unbeschränkt.

$$= \underbrace{\int_0^1 -x\,dx}_{(a)} + \underbrace{\int_1^2 x-2\,dx}_{(b)} + \underbrace{\int_2^\infty 0\,dx}_{(B)}$$

$$= -\left[\tfrac{1}{2}x^2\right]_0^1 + \left[\tfrac{1}{2}x^2 - 2x\right]_1^2 = -\tfrac{1}{2} + (-2) - \left(-\tfrac{3}{2}\right) = \boxed{-1}$$